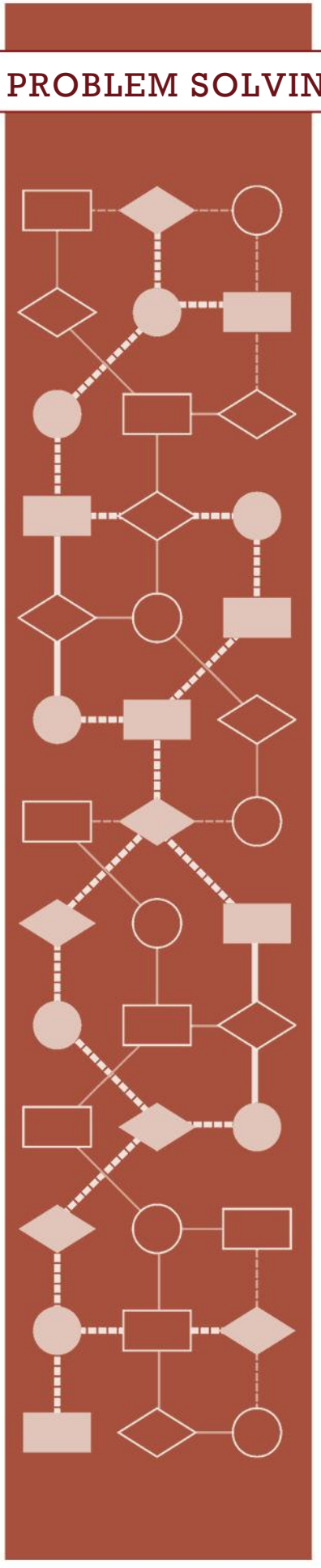
PROBLEM SOLVING



Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Comparing Data

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Comparing Data

MATHEMATICAL GOALS

This lesson unit is intended to help students to make meaningful comparisons between sets of data. In particular, students will develop their abilities in the following areas:

- Selecting appropriate measures of center and variability in order to summarize the important features of a set of data.
- Use quantitative measures to justify an argument.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7.SP Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.

INTRODUCTION

This lesson is structured in the following way:

- Before the lesson, students work individually on an assessment task, *Getting James to Work*, designed to reveal their current understandings and difficulties. You review their solutions and create questions for them to answer in order to improve their work.
- The lesson begins with a whole-class introduction in which students review important terms.
- Students work in pairs on a collaborative task in which they create data to satisfy written descriptions. When they have constructed their data sets, they separate their data from the descriptive text and pass the data to students in another group, who have not worked with the data to construct a description of their own. The two descriptions are then compared.
- In a whole-class discussion students discuss their approaches to the task and what they have learned.
- In a follow-up lesson, students receive your comments on the assessment task and use these to attempt another task, approaching it with insights that they have gained from the lesson.

MATERIALS REQUIRED

Each student will need a copy of the assessment tasks *Getting James to Work* and *Running Times*, a mini-whiteboard, pen and eraser.

Each small group of students will need either the sheet *Getting Raj to Work (1)* or the sheet *Getting Raj to Work (2)*, blank paper, calculators and a pair of scissors. Extension tasks *Getting Raj to Work – Extension (1)/(2)* are also available for use after the lesson if desired.

TIME NEEDED

20 minutes before the lesson for the assessment task, a 110-minute lesson (or two 55-minute lessons) and 30 minutes in a follow-up lesson (or for homework). Timings given are approximate and will depend on the needs of your class.

BEFORE THE LESSON

Assessment task: *Getting James to Work* (20 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the follow-up lesson.

Give each student a copy of the assessment task *Getting James to Work*.

Briefly introduce the task, helping the class to understand the problem. You could ask:

How did you get to school this morning? [E.g. bus, car, train, walk, etc.]

Is there more than one feasible way or do you not really have any choice?

Depending on your school's location, some options may be impossible or unlikely and you may have a lot of or very little variety among the students in your class.

Getting James to Work

James wants to get to work as quickly and reliably as possible in the mornings.

He tries three different transport methods:

- cycle all the way
- drive all the way
- walk to the railway station, take the train, and walk from the station.

He tries each method several times and records how many minutes the entire journey takes:

bicycle	28	24	25	29	25	26	26	23	29	25
car	19	21	32	57	31	27	21	24		
walk-train-walk	21	24	31	26	24	30				

Look carefully at James' results.

1. Use the data to make a case for why he should travel to work by bicycle.

2. Use the data to make a case for why he should travel to work by car.

James has got three options for getting to work. He has written down how long each one took him.

How many times did he try going by car? [8] How did you figure that out?

These questions are intended to get students examining the data and to help them to see that each number represents a day.

Use what you notice about the data and the meaning of the numbers listed for bicycles, car and walk-train-walk to answer the questions on the sheet.

It is important that, as far as possible, students answer the questions without assistance. If students are struggling to get started, ask questions that help them understand what they are being asked to do, but do not do the problem for them. The first few questions on the *Common issues* table may be helpful.

Students should not worry too much if they cannot understand or do everything, because there will be a lesson related to this, which should help them. Explain to students that by the end of the next lesson they should expect to answer questions such as these confidently; this is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem-solving approaches.

We suggest that you do not score students' work. Research suggests that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics. Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page.

We recommend that you:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions, and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students, and write these questions on the board when you return the work to the students.

The students will consider your comments when their work is returned to them in the follow-up lesson.

Common issues	Suggested questions and prompts
Does not refer to the data For example: The student writes that James should go by bicycle because it is cheaper/better for the environment.	• Can you use the data given on the sheet to make your case?
Assumes larger numbers are better For example: The student writes that going by car is best because it has the highest mean.	• Which is better for James: a higher or a lower mean? Why is this?
Calculates one measure for each method of transport For example: The student just uses the mean to support their case.	• Can you now consider other measures to support your case?
Refers to only one method of transport For example: The student states that James should go by bicycle because the mean time is 26 minutes.	• How does this compare with the other methods of transport?
Ignores the outlier in the car data For example: The student writes that the mean time by car is 29 minutes.	• One car day was very different from all the others. Which one? What could have caused this? What should we do about this value?
Makes a technical error For example: The student makes an arithmetic mistake when calculating measures.	• Does your answer seem reasonable? How could you check your answers?
Provides two good justifications For example: The student justifies travelling by bicycle and by car by making sensible comparisons.	• What other factors, apart from this data, might be important for James to consider when deciding how to get to work?

SUGGESTED LESSON OUTLINE

Whole-class introduction (10 minutes)

Give each student a mini-whiteboard, pen and eraser.

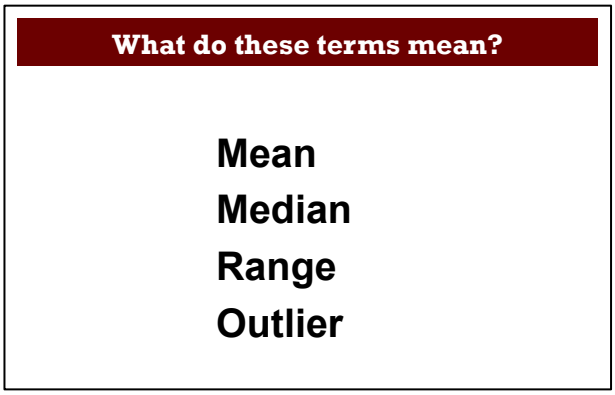
Briefly remind students about the *Getting James to Work* task.

Do you remember the Getting James to Work task? What was it about?

I'm not returning your work to you just yet. That will happen after this lesson.

We are going to be doing something similar today and will be using ideas like mean, median, range and outlier.

Display Slide P-1 showing a list of the four terms.



Ask students to say what they understand by each of them:

What do you know about these terms?

How are they useful in making comparisons?

If students are unsure of their meaning then they could discuss them with each other and write their answers on mini-whiteboards. They will need to be conversant with these terms for the rest of this lesson.

Collaborative small group work (1): constructing data (35 minutes)

Ask students to work in groups of two or three.

Give each group some blank paper, a pair of scissors and either *Getting Raj to Work (1)* or *Getting Raj to Work (2)*. (Trials of this unit suggest that students find completing the B1 data table more difficult than completing table A1 so you may wish to use this information when determining which task to give to which group.)

Provide students with a calculator if they do not have one.

Explain to students what they are to do:

Raj is travelling to work and he can go by bicycle or by car.

Your card has some data on it but there is some data missing.

At the top of the card someone has written a description of the data and a conclusion about whether Raj should travel to work by bicycle or by car.

Your task is to complete the data table so that it fits the description. You are going to use the information given to make up some times to go in the table.

You can use a calculator if you need to and there is blank paper available for rough work.

Display Slide P-2, which summarizes these instructions:

Collaborative work: constructing data

1. Use the information given to make up times to go in the table. The times are all in minutes.
2. Check that the data you create fits *all* of the information contained in the description at the top of the card.
3. Consider whether the numbers in the table are reasonable in the given context. Amend any that are not, ensuring that the card *still* fits the description.

While students are working, you have two tasks: to notice student approaches to the task and to support problem solving.

Make a note of student approaches to the task

Notice how students make a start on the task, where they get stuck and how they respond if they do come to a halt. Do students start with one component of the description, and if so, which measure do they address first? Or do they start by inventing data and hope that it will work? Do they assume that the table must be completed with ten values to match the number already given for the bicycle/car or do they invent less/more data values? Do they write their made-up values in order or do they rearrange the data to check it meets the criteria given in the description? Do they comment on the conclusion of whether Raj should travel by bicycle or by car? You can use this information to focus a whole-class discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that push students towards a particular approach to the task. Instead, ask questions to help students clarify their thinking.

The following questions and prompts may be helpful:

Can you check the data that has been given to you on the card? Does it fit the description?
Which fact in the description are you going to begin with? Why?
Which facts in the description are you going to leave until later? Why?

If you find students are unproductively struggling for some time you may want to suggest they:

- Find data values that approximately fit the descriptions. This is good enough.
- Or/and create fewer than ten data values.

If students still do not know where to begin, to help them to develop useful strategies you may want to ask:

Can you write down five numbers with a mean of 10?
Can you write down five numbers with a mean of 10 and a median that isn't 10?
Can you write down five numbers with a mean of 10 and a median of 10 and one outlier?
Can you write down five numbers with a mean of 10 and a range of 3?

The figures five and ten have been carefully chosen because they are easy numbers to use in division and multiplication. Also an odd number is preferable when calculating the median.

Probe students for the strategies they used to obtain the data values in these simpler cases and see if they can apply these approaches to the information given on the card.

If students are making good progress with the task, encourage them to check that their data values are reasonable in the given context:

Your values may fit the description, but are they realistic?

Can you now change some of them to make them more realistic (but still fit the description)?

Collaborative small group work (2): writing descriptions (25 minutes)

When most groups have finished completing their card and you judge that an appropriate amount of time has been spent on the task, stop the class:

Take your scissors and cut along the dashed line.

*Swap the **data table** with another group. If you have card A1, swap it with another group's card B1 and vice versa. Keep the top part!*

Your task now is to write a description of the data that you have just been given. You must refer to the mean, median and range in your description and comment on any outliers.

Try to reconstruct what is written at the top of the card! It won't be exactly the same, but it will be interesting to see the similarities and differences.

Once you have completed your description, explain whether you think Raj should travel to work by bicycle or by car.

Students should write their description on a piece of paper. They can use the top parts of their own cards to remind them of the kinds of things that they might write.

Slide P-3 summarizes these instructions:

Collaborative work: writing descriptions

1. Cut along the dashed line.
2. Swap the data table with another group (A1s swap with B1s).
3. Write a description of the data you have just been given. Refer to the mean, median and range and comment on any outliers. (It may not match exactly what it said on that card!)
4. Explain whether you think Raj should travel to work by bicycle or by car.

Collaborative small group work (3): comparison of descriptions (20 minutes)

Once the students have written their descriptions and come to a conclusion about which mode of transport Raj should use, they should pair up with the group whose data they were using and compare what they have written with the descriptions on the card.

See what the similarities and differences are between what you wrote and what it said on the card. Maybe what you wrote was better?

Check whether you came to the same conclusion about which mode of transport Raj should use.

Some differences will be due to focusing on different aspects of the data; others may be due to errors on the part of the group constructing the data. Encourage students to identify errors and think about any revisions that may be required.

Whole-class discussion (20 minutes)

In a whole-class discussion, depending on how the lesson went, encourage students to talk about what they have learned, strategies they used and/or what differences arose during the comparison of descriptions and conclusions.

Questions you might like to ask about strategies used to create data sets:

When figuring out the data values, what measure did you work with first? Why was that? Then what did you do?

Did anyone use a different strategy?

What difficulties did you encounter? Did you overcome these difficulties? How?

Questions you might like to ask about writing and comparing descriptions:

How close were your descriptions to the ones on the card?

Did you come to the same conclusions?

Why do you think that was?

What were the differences? Were they significant? Why?

Follow-up lesson: reviewing the assessment task (30 minutes)

Begin by returning to the students the initial assessment task *Getting James to Work*. If you have chosen not to write questions on individual student papers, display your list of questions on the board.

Here are my comments on the work you did [a few days ago]. Working individually, consider your responses to my questions and how you could improve your work. Write your responses on the back if there isn't space on the task sheet.

Give students a copy of the task *Running Times*.

Now, see if you can use what you have learnt last lesson to complete this similar task.

Teachers may prefer to give this as a homework task.

Extension task

If you feel that your class needs further practice at this kind of activity, then two extension cards have also been provided: *Getting Raj to work - Extension (1) and (2)*. These are more open and students find them more difficult.

SOLUTIONS

Assessment task: *Getting James to Work*

The statistics in the table below may help you to interpret students' answers:

	Bicycle	**Car**	**Walk-train-walk**	**Car** (*with outlier omitted*)
Mean	26	29	26	25
Median	25.5	25.5	25	24
Range	6	38	10	13
Standard Deviation *(correct to 2 decimal places)*	2.05	12.27	3.85	5.13

Given the small amount of data and the similarity in the mean values, students may feel that there is not much difference and little basis for a firm conclusion. These are important issues for them to consider.

The following comments on each question are for guidance only:

1. The case for going by bicycle could draw on the fact that the mean is the (joint) smallest, coupled with the fact that the spread of the data is less. This means that James can be more confident of the time that it will take him to get to work, as the values are more consistent.

   Students might also comment on the cheaper cost and environmental and fitness benefits of cycling, the ease with which he can speed past traffic jams and the possibility that after weeks of cycling to work he may be able to make the journey even more quickly, as he gets fitter. These are not arguments based on the data, however.

2. The case for going by car initially looks weak, as the mean is higher than for the other modes of transport and the data is also spread out, making the journey time very variable and this method quite unreliable. However, much of this is caused by the one outlier of 57 minutes. Perhaps this was caused by a freak traffic jam? Without knowing how often such events occur, students might be unsure what to do with this data item. Omitting it gives the results in the right-hand column of the table above, giving car travel the smallest mean time. However, there remains the un-quantified risk of the occasional very long journey time. Even though there is no clear-cut rule about what to do with outliers, students should be aware of the problem and should certainly comment on a clear outlier such as this.

Collaborative small-group work

Possible data values are shown in the table below. Other values would also work. The mean, median and range are also shown in the table:

	Mode of transport	**Data** *(values in italics already given)*										**Mean**	**Median**	**Range**
A1	Bicycle	*25*	*22*	*26*	*23*	*28*	*23*	*25*	*27*	*24*	*27*	**25**	***25***	***6***
	Car	21	21	22	22	22	24	26	26	26	40	**25**	***23***	**19**
B1	Bicycle	12	12	13	13	25	25	25	25	25	25	**20**	**25**	**13**
	Car	*20*	*24*	*24*	*20*	*18*	*24*	*20*	*16*	*20*	*24*	**21**	**20**	***8***

If students have made minor calculation errors, or have managed to satisfy only some of the conditions in the descriptions, it is important to value what they have achieved and learned from the task, even if they have not completed it perfectly.

Extension task

	Mode of transport	**Data** *(values in italics already given)*										**Mean**	**Median**	**Range**
A2	Bicycle	18	18	19	19	20	20	20	20	21	25	**20**	**20**	**7**
	Car	14	16	19	22	22	24	25	25	26	27	**22**	**23**	**13**
B2	Bicycle	17	18	19	23	24	26	27	28	29	29	**24**	**25**	**12**
	Car	24	24	25	25	25	25	31	36	37	48	**30**	**25**	**24**

Assessment task: *Running Times*

The statistics in the table below may help you to interpret students' answers:

	Mary	**David**	**Sally**	**David** (*with outlier omitted*)
Mean	63	64	64	62
Median	65	62	64.5	62
Range	12	27	3	5
Standard Deviation *(correct to 2 decimal places)*	3.98	7.15	1.25	1.84

1. The case for not entering Sally into the race could draw on the fact that the mean is the (joint) highest, coupled with the fact that the median is the second highest out of the 3 runners. The spread of the data is less than the other two runners, suggesting that Sally is running consistently

at these times. Comparing Sally's race time with David's (who has a comparable mean), even when the outlier in David's data is removed, Sally's running times are still more consistent.

Students might also comment on the fact that there is less data for Sally than the other two runners. They may conclude that Sally is unreliable and so should not be entered into the race. This argument is not, however, based explicitly on the data.

2. The case for not entering Mary into the race may initially look weak, as the mean is lower than for the other runners and the times are more consistent than David's (although not as consistent as Sally's). However, the very high range of running times for David is caused by the one outlier of 86 minutes. Perhaps David was not feeling well on this occasion or injured himself during the training session? Without knowing how often such events occur, students might be unsure what to do with this data item. Omitting it gives the results in the right-hand column of the table above, giving David the smallest mean time, bettering Mary's mean of 63. However, there remains the un-quantified risk of the occasional very long run time. Even though there is no clear-cut rule about what to do with outliers, students should be aware of the problem and should certainly comment on a clear outlier such as this.

Getting James to Work

James wants to get to work as quickly and reliably as possible in the mornings.

He tries three different transport methods:

- cycle all the way
- drive all the way
- walk to the railway station, take the train, and walk from the station.

He tries each method several times and records how many minutes the entire journey takes:

bicycle	28	24	25	29	25	26	26	23	29	25
car	19	21	32	57	31	27	21	24		
walk-train-walk	21	24	31	26	24	30				

Look carefully at James' results.

1. Use the data to make a case for why he should travel to work by bicycle.

2. Use the data to make a case for why he should travel to work by car.

Getting Raj to Work (1)

A1

Description:

The mean times by car and by bicycle are the same.

The median car time is 23 minutes, whereas the median bicycle time is 25 minutes.

The range of bicycle times is 6 minutes, whereas the range of car times is only 5 minutes, if you exclude the outlier.

Conclusion:

Raj should go by car.

Use the description to complete the data table below:

A1

Bicycle times	25	22	26	23	28	23	25	27	24	27
Car times										

Getting Raj to Work (2)

B1

Description:

The mean time by bicycle is 1 minute less than the mean time by car.

The median time by bicycle is 5 minutes more than the median time by car.

The range of car times is 8 minutes, which is 5 minutes less than the range of bicycle times.

Conclusion:

Raj should go by car.

Use the description to complete the data table below:

B1

Bicycle times										
Car times	20	24	24	20	18	24	20	16	20	24

Running Times

An athletics coach is training three runners Mary, David and Sally to compete in a 10-kilometer race.

He can only enter two of the runners and needs to decide which runner will not be entered.

He knows how many minutes it has taken each runner to run 10 kilometers in their training sessions:

Mary	70	58	58	65	59	60	66	59	65	65	66	65
David	62	60	60	64	86	64	64	61	59	62	64	62
Sally	65	63	62	64	62	65	65	65	65	64		

Look carefully at the coach's results.

1. Use the data to make a case for why he should not enter Sally into the race.

2. Use the data to make a case for why he should not enter Mary into the race.

Getting Raj to Work - Extension (1)

A2

Description:

The mean is 2 minutes less by bicycle.

The median is 3 minutes less by bicycle.

The range by bicycle is also less, so travelling by bicycle is more consistent from day to day.

Conclusion:

Raj should go by bicycle.

Use the description to complete the data table below:

A2

Bicycle times	
Car times	

Getting Raj to Work - Extension (2)

B2

Description:

The mean time by bicycle is four-fifths of the mean time by car.

There is an outlier in the car times, but even if you exclude that, the mean time by bicycle is 4 minutes less.

The median times by car and by bicycle are the same.

Conclusion:

Raj should go by car.

Use the description to complete the data table below:

B2

Bicycle times	
Car times	

What do these terms mean?

Mean

Median

Range

Outlier

Collaborative work: constructing data

1. Use the information given to make up times to go in the table. The times are all in minutes.

2. Check that the data you create fits *all* of the information contained in the description at the top of the card.

3. Consider whether the numbers in the table are reasonable in the given context. Amend any that are not, ensuring that the card *still* fits the description.

Collaborative work: writing descriptions

1. Cut along the dashed line.

2. Swap the data table with another group (A1s swap with B1s).

3. Write a description of the data you have just been given. Refer to the mean, median and range and comment on any outliers. (It may not match exactly what it said on that card!).

4. Explain whether you think Raj should travel to work by bicycle or by car.

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
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It was refined on the basis of reports from teams of observers led by
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along with comments from teachers and other users.

This project was conceived and directed for
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