CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Evaluating Statements about Enlargements (2D & 3D)

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Beta Version

Evaluating Statements About Enlargements (2D & 3D)

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to solve problems involving area and volume, and in particular, to help you identify and assist students who have difficulties with the following:

- Computing perimeters, areas and volumes using formulas.
- Finding the relationships between perimeters, areas, and volumes of shapes after scaling.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-GMD Geometric Measurement and Dimension:

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

3. Construct viable arguments and critique the reasoning of others.
7. Look for and make use of structure.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understandings and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- After a whole-class introduction, students work in small groups. They consider the results of doubling the measurements of a range of shapes in two and three dimensions.
- After a whole-class discussion, students return to their original task and try to improve their own responses.

MATERIALS REQUIRED

- Each individual student will need two copies of the assessment task *A Fair Price*, and a mini-whiteboard, a pen, and an eraser
- Each small group of students will need *Card Set: Scaling Up* (cut up into cards), the *Formula Sheet*, a large sheet of paper, and a glue stick.
- Calculators should be available for students who request them.
- There are some projector resources to help with whole-class discussions.

TIME NEEDED

15 minutes before the lesson, and a single 70-minute lesson (or two 40-minute lessons). Timings given are approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *A Fair Price* (15 minutes)

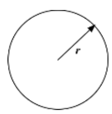
Set this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work to find out the kinds of difficulties students have with it. Then you will be able to target your help more effectively in the follow-up lesson.

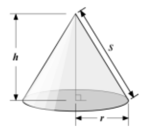
Give each student a copy of *A Fair Price.*

A Fair Price

In the following questions, *fair price* means that the amount you get is in proportion to the amount you pay. For example, the fair price for twelve cookies is double the cost of six.

You may find the following formulae useful.





Area of a circle: πr^2

Circumference of circle: $2\pi r$

Volume of a cone: $\frac{1}{3}\pi r^2 h$

1. Candy Rings

A large ring of candy has a diameter of 8 inches, and a small ring has a diameter of 4 inches.

(Diagram not to scale.)

Jasmina says:

"I get the same amount of candy from two small rings as from one large ring."

1. Is Jasmina correct? If you think Jasmina is correct explain why. If you think she is incorrect, replace the statement with one that is correct. Explain why your statement is correct.

....

If the price of the small ring of candy is 40 cents, what is a *fair price* for a large one? Explain your answer.

....

2. Pizzas

A large pizza has a diameter of 12 inches. A small pizza has a diameter of 6 inches.

(Diagram not to scale.)

I get the same amount of pizza from three small pizzas as from one large pizza.

Is Jasmina correct about the pizzas?

If you think Jasmina is correct explain why.

If you think she is incorrect replace the statement with one that is correct. Explain why your statement is correct.

....

If the price for a small pizza is $3, what is a *fair price* for a large one?

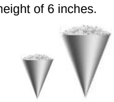
Explain your answer.

....

3. Popcorn

The larger cone has a top radius of 4 inches and a height of 12 inches.

The small cone has a top radius of 2 inches and a height of 6 inches.



(Diagram not to scale.)

"I get the same amount of popcorn from two small cones as from one large cone."

Is Jasmina correct about the popcorn cones?

If you think Jasmina is correct, explain why.

If you think she is incorrect, replace the statement with one that is correct. Explain why your statement is correct.

....

If the price for a small cone of popcorn is $1.20, what is a *fair price* for a large one?

....

Briefly introduce the task and help the class to understand the problems and their context.

Read through the questions and try to answer them as carefully as you can.

Show all your work so that I can understand your reasoning.

Explain to the students what 'fair price' means.

In the questions, the term 'a fair price' means that the amount you get should be in proportion to the amount you pay.
So for example, if a pound of cookies costs $3, a fair price for two pounds will be $6.

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything because, in the next lesson, they will engage in a similar task that should help them. Explain to students that by the end of the next lesson, they should expect to be able to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores, and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given below. These have been drawn from common difficulties observed in trials of this unit.

We suggest that you write a list of your own questions, based on your students' work, using the ideas that follow. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will be of help the majority of students. These can be written on the board at the end of the lesson.

Write a list of questions, applicable to your own class. If you have enough time, add appropriate questions to each piece of your students' work.

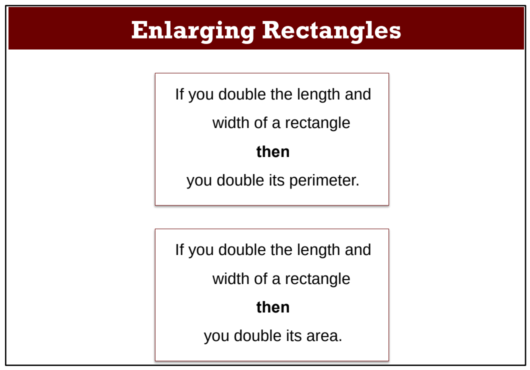
Common issues:	**Suggested questions and prompts:**
Student assumes the diagrams are accurate representations For example: The student writes "I've counted the candy. The larger circle has more than twice the amount of candy that the smaller one has." Or: The student writes "Three small pizzas fit into the large one."	• The pictures are not accurate. • How can you use math to check that your answer is accurate?
Student fails to mention scale For example: The areas of the two pizzas are calculated but not the scale of increase.	• How can you figure out the scale of increase in area/volume using your answers?
Student focuses on non-mathematical issues For example: The student writes "It takes longer to make three small pizzas than one large one. The large one should cost $8."	• Now consider a fair price from the point of view of the customer. • Are three small pizzas equivalent to one big one? How do you know?
Student makes a technical error For example: The student substitutes the diameter into the formula instead of the radius. Or: The student makes a mistake when calculating an area or volume.	• What does r in the formula represent? • Check your calculations.
Student simply triples the price of the pizza or doubles the price of a cone of popcorn.	• Do you really get three times as much pizza? • Do you really get twice as much popcorn?
Student correctly answers all the questions Student needs an extension task.	• If a pizza is made that has a diameter four times bigger (ten times/n times), what should its price be? How do you know? Can you use algebra to explain your answer? • If a cone of popcorn has a diameter and height four times bigger (ten times/n times), what should its price be? How do you decide? Can you use algebra to explain your answer?

SUGGESTED LESSON OUTLINE

If you have a short lesson or you find the lesson is progressing at a slower pace than anticipated, then you may want to spend two lessons on the tasks. We give suggestions on how to manage this below.

Whole-class introduction (10 minutes)

This introduction will provide students with a model about how they should work during the collaborative tasks. Give each student a mini-whiteboard, a pen, and an eraser. Use the projector resource *Enlarging Rectangles*.



Decide whether each statement is true or false.

Write a convincing explanation.

If you think a statement is false then replace it with a correct one.

After a few minutes ask two or three students for their answers. Encourage them to write their explanations on the board. If students are struggling to provide convincing arguments, you could ask the following questions:

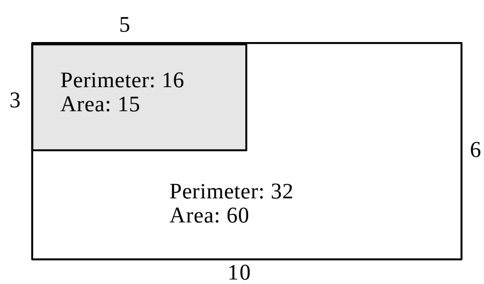
Can you use a diagram to convince me? Show me.

How do you know for sure your answer is correct for all rectangles?

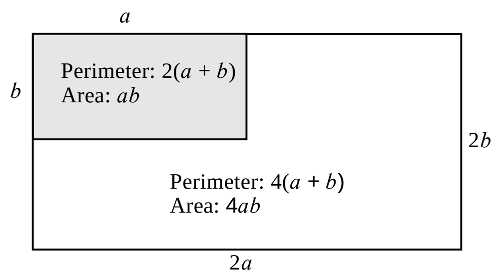
Can you use algebra to convince me? Show me.

If the area is not doubled, then what scaling is taking place? Why is this?

Students may find it easiest to start by considering specific examples.



Encourage students to consider the statements more generally.



In this way, students may see that the first statement is true, but the second should be revised.

If you double the length and width of a rectangle, then you multiply its area by 4.

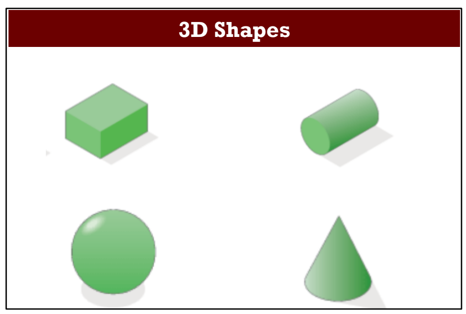
You may then want to look at different scale factors:

If the two measurements are multiplied by 3 instead of doubled what happens to the perimeter and area?

If the two measurements are increased by a scale factor of 10/n, what happens to the area?

Collaborative activity: *Scaling Up* (20 minutes)

To introduce this task you may want to use the projector resource *3D Shapes*.



You could also show the class real examples of these 3D shapes.

Organize the students into groups of two or three.

Give each group the cut-up cards *Scaling Up,* a copy of the *Formula Sheet,* a large sheet of paper, and a glue stick.

The cards show rectangular prisms, circles, spheres, cylinders, and cones.

Sort the cards into these five different mathematical objects.

Your task is to decide whether each statement is true or false.

If you think a statement is false, change the second part of the statement to make it true.

Try to figure out what it is about the formula for the shape's area or volume that makes the statement true or false.

Show calculations, draw diagrams, and use algebra to convince yourself that you have made a correct decision.

When everyone in your group agrees with the decision for one object, place the statements on the poster and write your explanations around it.

Begin by working with the statements on rectangular prisms.

You may not have time to consider all twelve statements.

It is better for you to explain your reasoning fully for a few statements than to rush through trying to decide whether all the statements are true or false.

If you think your class will understand the notion of leaving π in the answer, then do not give out calculators as their use can prevent students noticing the factor of the increase.

You have two tasks during the small group work: to make a note of student approaches to the task, and to support student problem solving.

Make a note of student approaches to the task

Listen and watch students carefully. In particular, listen to see whether they are addressing the difficulties outlined in the *Common issues* table (above). You can use this information to focus a whole-class discussion towards the end of the lesson.

Support student problem solving

Try not to do the thinking for students, but rather help them to reason for themselves. Encourage students to engage with each others' explanations, and take responsibility for each others' understanding.

> *Judith, why do you think this statement is true/false?*
>
> *James, do you agree with Judith? Can you put her explanation into your own words?*

If students are struggling to get started on the task:

> *What formula can you use to check if the statement is correct? What values can you put into this formula?*
>
> *If it is not twice as big, by what factor has the area/volume increased? How do you know?*

At first, you may want to focus your questioning on the cards about rectangular prisms:

> *Are any of these statements true? What is it about the formula that makes the formula true?*
>
> *What has the volume of the rectangular prism increased by for this statement? How does this increase relate to the formula?*

Students often prefer to multiply out π. This means they may not notice the factor of increase.

> *Can you express these two areas/volumes for this statement as multiples of π? How does this help?*

Students often do not recognize the relationship between the formula and the factor of the increase.

> *Show me two statements that are correct. What has doubled in each formula? What has remained the same?*
>
> *Show me two statements where the area or volume has increased by a factor of four. Look at the two formulas and figure out why the area(s)/volume(s) has increased by the same factor.*
>
> *Show me two statements where just the radius is doubled but the factors of increase are different. Look at the two formulas and figure out why the area(s)/volume(s) has increased by a different factor?*
>
> *When the radius/height of this shape is doubled, what variable will change in the formula? [E.g., r, r^2,* or *h.] How does this affect the area/volume?*

If a lot of students are struggling on the same issue you may want to hold a brief whole-class discussion.

Encourage students who work through the task more quickly to think about how they can explain the scaling in general terms. They may use algebra in their explanation, or simply highlight the properties of a formula that determine the scaling.

Can you use algebra to show you are correct? If the radius has a length of n, what is double its length? How can you use this in the formula?

Can you figure out if the statement is true or not just by looking at the formula? Why? Why not?

If students finish early, have them consider what happens if the phrase 'multiply by 3' replaces the word 'double.'

Extending the activities over two lessons

You may decide to spread the work over two lessons. If so, ask students to stop working on the task 10 minutes before the end of the lesson.

Ask students to glue the cards that they have worked on to the large sheet of paper. Remind them that there should be an explanation accompanying each card. Students can then use a paperclip to attach any remaining cards to their posters.

Hold a short whole-class discussion. Ask a representative from each group to use their group poster to explain their thinking about one statement to the whole-class. Encourage the rest of the class to challenge their explanations, but avoid intervening too much yourself.

You can then re-start the lesson with more poster work or by sharing posters (immediately below), as you see fit.

Sharing posters (10 minutes)

When a group has completed all the statements about one object, ask the students to compare their reasoning with that of a neighboring group.

Check which answers are different.

A member of each group needs to explain their reasoning for these answers. If anything is unclear, ask for clarification.

Then together consider if you should change any of your answers.

It is important that everyone in both groups understands the math. You are responsible for each other's learning.

Whole-class discussion (20 minutes)

Discuss as a class how the structure of a formula determines the increase.

Find me a card where the statement is correct. How does the formula relate to an increase by a factor of two? Find me another. What do the formulas have in common?

Find me a card that uses a formula involving r^2. *Is the statement correct? Why?/Why not? By what factor has the area/volume increased? Is it the same increase for all cards that use* r^2*? Why/Why not?*

Find me a card that uses a formula involving r^3. *Is the statement correct? Why?/Why not? By what factor has the volume increased?*

Now try to extend some of these generalizations. Use the projector resource *Is it correct?*

Is it correct?

1.
If you treble the length and width of a rectangle
then
the perimeter increases by a factor of 3.

2.
If you treble the length and width of a rectangle
then
the area increases by a factor of 6.

3.
If you treble the length, width and height of a rectangular prism
then
the volume increases by a factor of 9.

I want you to decide if any of these statements are true.

If you think a statement is not true, then change the last part of the statement to make it true.

Students may have difficulties making decisions without using specific dimensions. Encourage those students who progressed well in the lesson to think of a general explanation.

If the statement is correct, how do you know?

If the statement is not correct, then what factor does the perimeter/area /volume increase by? How do you know?

How does the increase relate to the formula?

*If all the dimensions increase by five/ten/*n *times what happens to the perimeter/area and volume?*

Ask two or three students to explain their answer.

Some students will begin to see that for similar shapes the area scale factor is the square of the scale, i.e. 3^2, 5^2, 10^2, or n^2 and the volume scale factor is the cube of the scale, i.e. 3^3, 5^3, 10^3, or n^3.

If you have time you may want to consider the same task but use a different shape, such as a circle, cylinder, or cone.

Improving individual solutions to the assessment task (10 minutes)

Return to the students their original assessment *Fair Price*, as well as a second blank copy of the task.

Look at your original responses and think about what you have learned this lesson.

Using what you have learned, try to improve your work.

If you have not added questions to individual pieces of work then write your list of questions on the board. Students are to select from this list only the questions they think are appropriate to their own work.

If you find you are running out of time, then you could set this task as homework.

SOLUTIONS

ASSESSMENT TASK: *A FAIR PRICE*

1. Jasmina is correct: if the radius doubles so will the circumference. This is a linear relationship.

   The scale factor is 2.

   The circumference of the small candy ring is 4π.

   The circumference of the large candy ring is 8π.

   A fair price for the large candy ring is 80 cents, double the price of the small one.

2. Jasmina is incorrect.

   The area of a small pizza is 9π and that of a large one is 36π. The area of four small pizzas is equal to the area of one large pizza.

   The areas of similar figures are related by the square of the scale factor: $2^2 = 4$.

   A correct statement might be, "I get the same amount of pizza from four small ones as one large one."

   A fair price for the large pizza would be $12, or 4 times $3.

3. Jasmina is incorrect.

   The volume of the small cone is 8π and that of the large cone is 64π. The scale factor of the radius and height is 2.

   Volumes of similar figures are related by the cube of the scale factor: $2^3 = 8$.

   A correct statement might be, "I get the same amount of popcorn from eight small ones as one large one."

   A fair price would be $8 \times \$1.20 = \$9.60$.

Collaborative activity: *Scaling Up*

Below are the general solutions obtained by reasoning algebraically. Students will also provide solutions using on specific value.

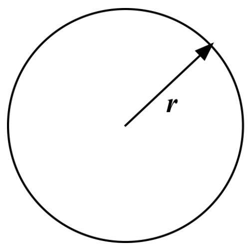
1. "If you double just the width of a rectangular prism then you double its volume" is **true**. If the width is *a*, $V = alh$ If the width is $2a$, $V = 2alh$ An increase by a factor of 2.	**2.** "If you double just the width and height of a rectangular prism then you double its volume" is **false**. If the width is *a* and the height is *b*, $V = alb$ If the width is $2a$ and the height is $2b$, $V = 4alb$. An increase by a factor of 4.

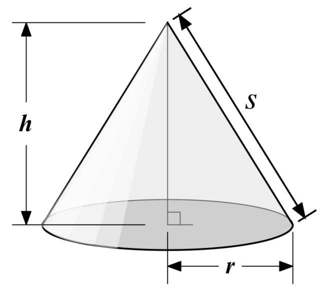
3. **"**If you double the width, height, and length of a rectangular prism then you double its volume" is **false**. If the width is a, the length is b and the height is c, $V = abc$ If the width is $2a$, the length is $2b$ and the height is $2c$, $V = 8abc$. An increase by a factor of 8.	**4.** **"**If you double the radius of a circle then you double its circumference" is **true.** If the radius is a, $C = 2\pi a$ If the radius is $2a$, $C = 4\pi a$ An increase by a factor of 2.
5. "If you double the radius of a circle then you double its area" is **false**. If the radius is a, $A = \pi a^2$ If the radius is $2a$, $A = 4\pi a^2$ An increase by a factor of 4.	**6. "**If you double the radius of a sphere then you double its surface area" is **false**. If the radius is a, $A = 4\pi a^2$ If the radius is $2a$, $A = 16\pi a^2$ An increase by a factor of 4.
7. "If you double the radius of a sphere then you double its volume" is **false**. If the radius is a, $V = \frac{4}{3}\pi a^3$ If the radius is $2a$, $V = \frac{32}{3}\pi a^3$ An increase by a factor of 8.	**8. "**If you double just the radius of a cylinder then you double its curved surface area" is **true.** If the radius is a, $A = 2\pi ah$ If the radius is $2a$, $A = 4\pi ah$. An increase by a factor of 2.
9. "If you double just the height of a cylinder then you double its volume" is **true.** If the height is b, $V = \pi r^2 b$ If the height is $2b$, $V = 2\pi r^2 b$ An increase by a factor of 2.	**10.** "If you double both the radius and height of a cylinder then you double its volume" is **false**. If the radius is a and the height is b, $V = \pi a^2 b$ If the radius is $2a$ and the height is $2b$, $V = 8\pi a^2 b$ An increase by a factor of 8.
11. "If you double just the base radius of a cone then you double its volume" is **false**. If the base radius is a, $V = \frac{1}{3}\pi a^2 h$ If the base radius is $2a$, $V = \frac{4}{3}\pi a^2 h$ An increase by a factor of 4.	**12. "**If you double both the height and base radius of a cone then you double its volume" is **false**. If the base radius is a, and the height is b, $V = \frac{1}{3}\pi a^2 b$ If the base radius is $2a$, and the height is $2b$, $V = \frac{8}{3}\pi a^2 b$ An increase by a factor of 8.

A Fair Price

In the following questions, 'fair price' means that the amount you get is in proportion to the amount you pay. For example, the 'fair price' for twelve cookies is double the cost of six.

You may find the following formulae useful.





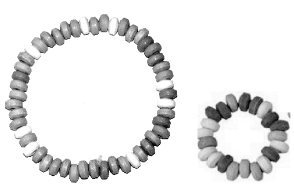
Area of a circle: πr^2

Circumference of circle: $2\pi r$

Volume of a cone: $\frac{1}{3}\pi r^2 h$

1**. Candy Rings**

A large ring of candy has a diameter of 8 inches, and a small ring has a diameter of 4 inches.



(Diagram not to scale.)

Jasmina says:

"I get the same amount of candy from two small rings as from one large ring."

1. Is Jasmina correct? If you think Jasmina is correct explain why. If you think she is incorrect, replace the statement with one that is correct. Explain why your statement is correct.

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If the price of the small ring of candy is 40 cents, what is a *fair price* for a large one? Explain your answer.

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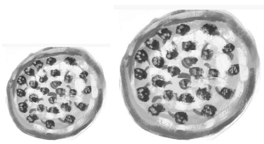
2. Pizzas

A large pizza has a diameter of 12 inches.
A small pizza has a diameter of 6 inches.



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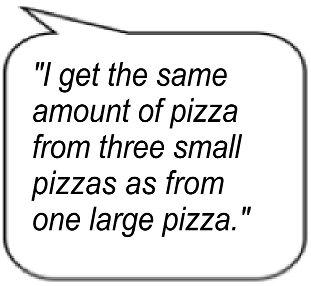

Is Jasmina correct about the pizzas?

If you think Jasmina is correct explain why.

If you think she is incorrect replace the statement with one that is correct. Explain why your statement is correct.

....

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....

If the price for a small pizza is $3, what is a 'fair price' for a large one?

Explain your answer.

....

....

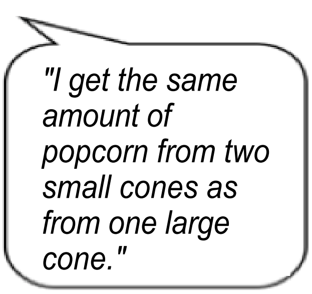
....

3. Popcorn

The larger cone has a top radius of 4 inches and a height of 12 inches.

The small cone has a top radius of 2 inches and a height of 6 inches.

(Diagram not to scale.)



Is Jasmina correct about the popcorn cones?

If you think Jasmina is correct, explain why.

If you think she is incorrect, replace the statement with one that is correct. Explain why your statement is correct.

....

....

....

If the price for a small cone of popcorn is $1.20, what is a 'fair price' for a large one?

....

....

....

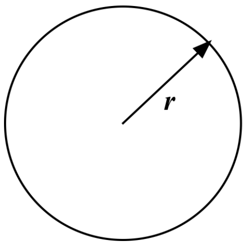
True or False?

1. If you double just the width of a rectangular prism then **you double its volume**.	2. If you double just the width and height of a rectangular prism then **you double its volume**.
3. If you double the width, height, and length of a rectangular prism then **you double its volume**.	4. If you double the radius of a circle then **you double its circumference**.
5. If you double the radius of a circle then **you double its area**.	6. If you double the radius of a sphere then **you double its surface area**.

True or False? (continued)

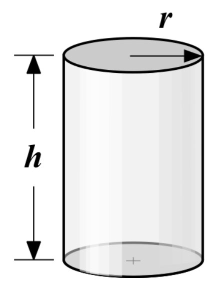
7. If you double the radius of a sphere then **you double its volume**.	8. If you double just the radius of a cylinder then **you double its curved surface area**.
9. If you double just the height of a cylinder then **you double its volume**.	10. If you double both the radius and height of a cylinder then **you double its volume**
11. If you double just the base radius of a cone then **you double its volume**	12. If you double both the height and base radius of a cone then **you double its volume**.

Formula Sheet



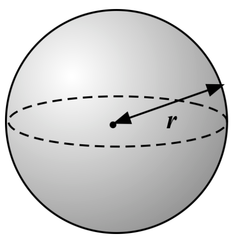
Area of a circle: πr^2

Circumference of a circle: $2\pi r$



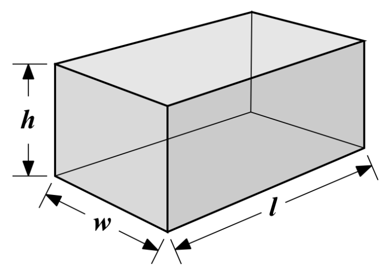
Volume of a cylinder: $\pi r^2 h$

Curved surface area of a cylinder: $2\pi rh$



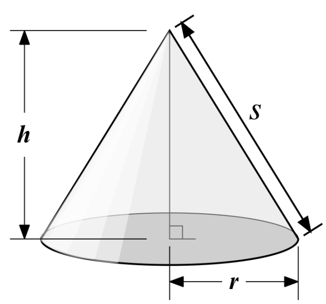
Volume of a sphere: $\frac{4}{3}\pi r^3$

Surface area of a sphere: $4\pi r^2$



Volume of a right rectangular prism: lwh

Surface area of a right rectangular prism: $2(lw + lh + wh)$



Volume of a cone: $\frac{1}{3}\pi r^2 h$

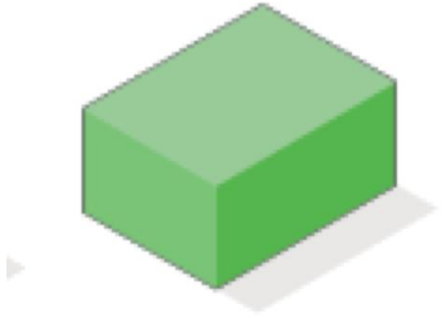
Curved surface area of a cone: πrs

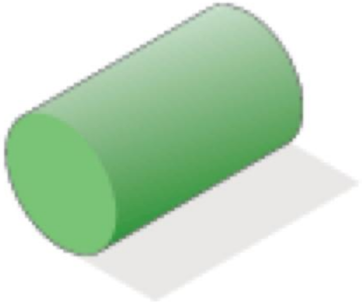
Enlarging Rectangles

If you double the length and width of a rectangle
then
you double its perimeter.

If you double the length and width of a rectangle
then
you double its area.

3D Shapes





True or False?

- If you think a statement is false, change the second part of the statement to make it true.
- Try to figure out what it is about the formula for the shape's area or volume that makes the statement true or false.
- Show calculations, draw diagrams, and use algebra to convince yourself that you have made a correct decision.
- When everyone in your group agrees with the decision for one object, place the statements on the poster and write your explanations around it.
- Begin by working with the statements on rectangular prisms.

Is it correct?

1.

If you treble the length and width of a rectangle
then
the perimeter increases by a factor of 3.

2.

If you treble the length and width of a rectangle
then
the area increases by a factor of 6.

3.

If you treble the length, width and height of a rectangular prism
then
the volume increases by a factor of 9.

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
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at the
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It was refined on the basis of reports from teams of observers led by
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based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
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