CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Forming Quadratics

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Forming Quadratics

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to understand what the different algebraic forms of a quadratic function reveal about the properties of its graphical representation. In particular, the lesson will help you identify and help students who have the following difficulties:

- Understanding how the factored form of the function can identify a graph's roots.
- Understanding how the completed square form of the function can identify a graph's maximum or minimum point.
- Understanding how the standard form of the function can identify a graph's intercept.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

A-SSE: Write expressions in equivalent forms to solve problems.

F-IF: Analyze functions using different representations.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2 Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

INTRODUCTION

The lesson is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understandings and difficulties. You then review their work, and create questions for students to answer in order to improve their solutions.
- After a whole-class interactive introduction, students work in pairs on a collaborative discussion task in which they match quadratic graphs to their algebraic representation. As they do this, they begin to link different algebraic forms of a quadratic function to particular properties of its graph.
- After a whole-class discussion, students return to their original assessment tasks and try to improve their own responses.

MATERIALS REQUIRED

- Each individual student will need two copies of the assessment task *Quadratic Functions* and a mini-whiteboard, pen, and eraser, or graph paper.
- Each pair of students will need *Domino Cards 1* and *Domino Cards 2*, cut horizontally into ten 'dominoes'.

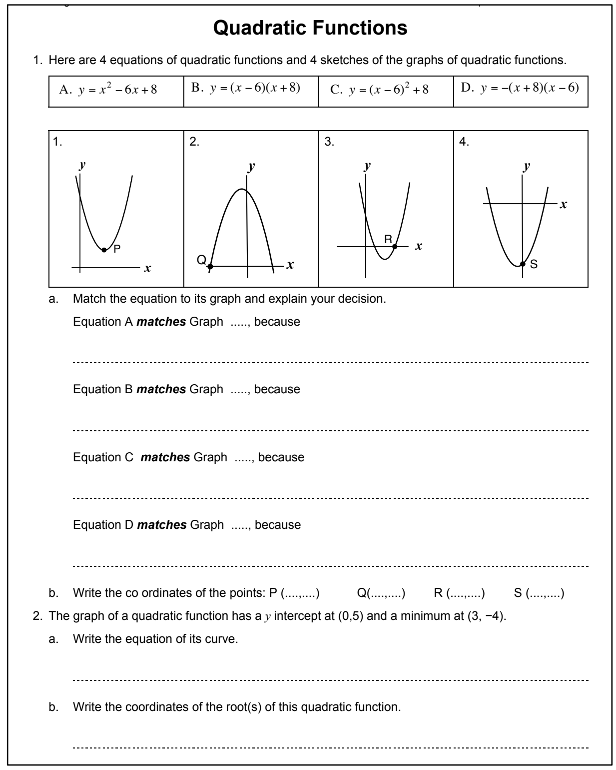
TIME NEEDED

15 minutes before the lesson, and 80-minute lesson (or two 40-minute lessons.) Timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Quadratic Functions* (15 minutes)

Have the students do this task in class or for homework, a day or more before the formative assessment lesson. This will give you an opportunity to assess the work, and to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.



Quadratic Functions

1. Here are 4 equations of quadratic functions and 4 sketches of the graphs of quadratic functions.

A. $y = x^2 - 6x + 8$	B. $y = (x-6)(x+8)$	C. $y = (x-6)^2 + 8$	D. $y = -(x+8)(x-6)$

a. Match the equation to its graph and explain your decision.

Equation A ***matches*** Graph ....., because

....

Equation B ***matches*** Graph ....., because

....

Equation C ***matches*** Graph ....., because

....

Equation D ***matches*** Graph ....., because

....

b. Write the co ordinates of the points: P (....,....) Q(....,....) R (....,....) S (....,....)

2. The graph of a quadratic function has a y intercept at (0,5) and a minimum at (3, −4).

a. Write the equation of its curve.

....

b. Write the coordinates of the root(s) of this quadratic function.

....

Give each student a copy of *Quadratic Functions*.

Briefly introduce the task and help the class to understand the problem and its context.

> *Read through the task and try to answer it as carefully as you can.*
>
> *Show all your work so that I can understand your reasoning.*

It is important that as far as possible, students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to be able to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive as it will encourage students to compare their scores, and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest that you write a list of your own questions, based on your students' work, using the ideas that follow. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will be of help to the majority of students. These can be written on the board at the end of the lesson.

Common issues	Suggested questions and prompts
Q1. Student has difficulty getting started	• You are given two pieces of information. Which form of a quadratic equation can you match this information to?
Q2. Student makes incorrect assumptions about what the different forms of the equation reveal about the properties of its parabola	• What does an equation in standard form tell you about the graph? Explain. • What does an equation in completed square form tell you about the graph? Explain.
Q2. Student uses an inefficient method For example: For each quadratic function, the student figures out the coordinates of several points by substituting x-values into the equation.	• Your method is quite difficult work. Think about the information each equation tells you about its graph. Think about the information each graph tells you about its equation.
Student makes a technical error For example: The student makes an error when manipulating an equation.	• Check your answer.
Student correctly answers all the questions **The student needs an extension task.**	• Q2. Can you think of any more coordinates for the key features of the Graphs 1, 2,3, and 4? Explain your answers. • Another quadratic has the same coordinates for the minimum, but the y-intercept is (0,14). What is the equation of this curve? [$y = 2x^2 - 12x + 14$]

SUGGESTED LESSON OUTLINE

If you have a short lesson, or you find the lesson is progressing at a slower pace than anticipated, then we suggest you end the lesson after the first collaborative activity and continue in a second lesson.

Whole-class interactive introduction: key features of quadratics (10 minutes)

Give each student either a mini-whiteboard, pen and eraser, or graph paper.

Introduce the lesson with:

Today, we are going to look at the key features of a quadratic curve.

On your mini-whiteboards, draw the x-and y-axis and sketch two quadratic curves that look quite different from each other.

Allow students to work for a few minutes and then ask them to show you their whiteboards.

Be selective as to which student you ask to explain his or her graphs. Look for two sets of curves in particular:

- one of which has a maximum point, the other a minimum;
- one of which one has two roots, the other one or none;
- that are not parabolas.

What makes your two graphs different?

What are the common features of your graphs?

Elicit responses from the class and try to keep your own interventions to a minimum. Encourage students to use mathematical terms such as roots, y-intercepts, turning points, maximum, minimum.

As students suggest key features, write them as a list on the board under the heading 'Key Features of a Graph of a Quadratic'.

Ask about turning points:

How many turning points does each of your graphs have? Is this turning point a maximum or minimum?

Can the curve of a quadratic function have more than one turning point/no turning points?

If all students have drawn graphs with minimums, ask students to draw one with a maximum.

Ask about roots:

How many roots does each of your graphs have?
Where are these roots on your curve?

Does anyone have a graph with a different number of roots?

How many roots can a quadratic have?

If all students have drawn graphs with two roots, ask a student to draw one with one or no roots.

Ask about y-intercepts:

Has anyone drawn a graph with different y-intercepts?

Do all quadratic curves have a y-intercept?

Can a quadratic have more than one y-intercept?

Write on the board these three equations of quadratic functions:

Standard Form:	Factored Form:	Completed Square Form:
1. $y = x^2 - 10x + 24$	2. $y = (x - 4)(x - 6)$	3. $y = (x - 5)^2 - 1$

Here are the equations of three quadratic functions.

Without performing any algebraic manipulations, write the coordinates of a key feature of each of their graphs.

For each equation, select a different key feature.

Explain to students they should use key features from the list on the board.

For example, students may answer:

Equation 1. The y-intercept is at the point (0,24). The graph has a minimum, because the coefficient of x is positive.

Equation 2. The graph has a minimum and has roots at (4,0) and (6,0).

Equation 3. The graph has a minimum turning point at $(5, -1)$

If students struggle to write anything about Equation 3, ask:

How can we obtain the coordinates of the minimum from Equation 3?

To obtain the minimum value for y, what must be the value of x? How do you know?

Equation 3 shows that the graph has a minimum when $x = 5$. This is because $(x - 5)^2$ is always greater than or equal to zero, and it takes a minimum value of 0 when $x = 5$.

What do the equations have in common? [They are different representations of the same function.]

Completed square form can also be referred to as vertex form.

Now write these two equations on the board:

4. $y = -(x + 4)(x - 5)$ 5. $y = -2(x + 4)(x - 5)$

What is the same and what is different about the graphs of these two equations? How do you know?

For example, students may answer:

- Both parabolas have roots at $(-4,0)$ and $(5,0)$.
- Both parabolas have a maximum turning point.
- Equation 2 will be steeper than Equation 1 (for the same x value Equation 2's y value will be double that of Equation 1).

Whole-class introduction to Dominos (10 minutes)

Organize the class into pairs. Give each pair of students cut-up 'dominos' A, E, and H from *Domino Cards 1* and *Domino Cards 2*.

Explain to the class that they are about to match graphs of quadratics with their equations, in the same way that two dominoes are matched. If students are unsure how to play dominos, spend a couple of minutes explaining the game.

The graph on one 'domino' is linked to its equations, which is on another 'domino'.

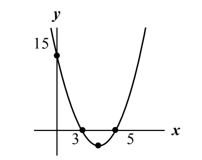
Place Card H on your desk. Figure out, which of the two remaining cards should be placed to the right of card H and which should be placed to its left.

A

$y = x^2 + 2x - 35$

....

....

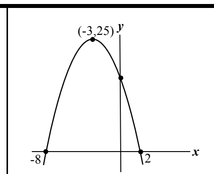


H

$y = x^2 - 8x + 15$

$y = (x - 3)(x - 5)$

$y = (x - 4)^2 - 1$

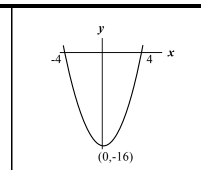


E

$y = -x^2 - 6x + 16$

$y = -(x + 8)(x - 2)$

$y = -(x + 3)^2 + 25$



Encourage students to explain **why** each form of the equation matches the curve:

Dwaine, explain to me how you matched the cards.

Alex, please repeat Dwaine's explanation in your own words.

Which form of the function makes it easy to determine the coordinates of the roots/ y-intercept/ turning point of the parabola?

Are the three different forms of the function equivalent? How can you tell?

The parabola on Domino A is missing the coordinates of its minimum. The parabola on Domino H is missing the coordinates of its y-intercept. Ask students to use the information in the equations to add these coordinates.

What are the coordinates of the minimum of the parabola on Card A? What equation did you use to work it out? [(4, −1)]

What are the coordinates of the y-intercept of the parabola on Card H? What equation did you use to work it out? [(0,16)]

At this stage, students may find it helpful to write what each form of the function reveals about the key features of its graph.

If you think students need further work on understanding the relationship between a graph and its equations, then ask students to make up three different algebraic functions, the first in standard form, the second in factored form, and the third in completed square form. Students are then to take these equations to a neighboring pair and ask them to explain to each other what each equation reveals about its curve.

Collaborative work: matching the dominos (15 minutes)

Give each pair of students all the remaining cut up *Domino Cards*.

Explain to students that the aim is to produce a closed loop of dominos, with the last graph connecting to the equations on 'domino' A. Students may find it easier to begin by laying the dominos out in a long column or row rather than in a loop.

You may want to use Slide P-1 of the projector resource to display the following instructions.

Take turns at matching pairs of dominos that you think belong together.

Each time you do this, explain your thinking clearly and carefully to your partner.

It is important that you both understand the matches. If you don't agree or understand ask your partner to explain their reasoning. You are both responsible for each other's learning.

On some cards an equation or part of an equation is missing. Do not worry about this, as you can carry out this task without this information.

You have two tasks during small-group work: to make a note of student approaches to the task, and support student problem solving.

Make a note of student approaches to the task

Notice how students make a start on the task, where they get stuck, and how they respond if they do come to a halt. You can use this information to focus a whole-class discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to this task. Instead, ask questions to help students clarify their thinking. If several students in the class are struggling with the same issue, write a relevant question on the board. You might also ask a student who has performed well on a particular part of the task to help a struggling student.

The following questions and prompts may be helpful:

Which form of the function makes it easy to determine the coordinates of the roots /y-intercept/turning point of the parabola?

How many roots does this function have? How do you know? How are these shown on the graph?

Will this function be shaped like a hill or a valley? How do you know?

Sharing work (5 minutes)

As students finish matching the cards, ask one student from each group to visit another group's desk.

If you are staying at your desk, be ready to explain the reasons for your group's matches.

If you are visiting another group, write your card matches on a piece of paper. Go to another group's desk and check to see which matches are different from your own.

If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.

When you return to your own desk, you need to consider as a pair whether to make any changes to your own work.

You may want to use Slide P-2 of the projector resource to display these instructions.

Collaborative work: completing the equations (15 minutes)

Now you have matched all the domino cards, I would like you to use the information on the graphs to fill in the missing equations and parts of equations.

You shouldn't need to do any algebraic manipulation!

Support the students as in the first collaborative activity.

For students who are struggling ask:

This equation is in standard form but the final number is missing. Looking at its graph, what is the value for y when x is zero? How can you use this to complete the standard form equation?

You need to add the factored form equation. Looking at its graph, what is the value for x when y is zero? How can you use this to complete the factored form equation?

Sharing work (5 minutes)

When students have completed the task, ask the student who has not already visited another pair to check their answers those of another pair of students. Students are to share their reasoning as they did earlier in the lesson unit.

Extension work

If a pair of students successfully completes the task then they could create their own dominos using the reverse side of the existing ones. To do this students will need to use algebraic manipulation to figure out all three forms of the function. Once students have written on all the dominos they should give them to another pair to match up. This is a demanding task so you may want to limit the number of dominos students use.

Whole-class discussion: overcoming misconceptions (10 minutes)

Organize a discussion about what has been learned. The intention is to focus on the relationships between the different representations of quadratic functions, not checking that everyone gets the right answers.

Ella, where did you place this card? How did you decide?

Ben, can you put that into your own words?

What are the missing equations for this graph? How did you work them out?

Did anyone use a different method?

Improving individual solutions to the assessment task (10 minutes)

Return to the students their original assessment *Quadratic Functions*, as well as a second blank copy of the task.

Look at your original responses and think about what you have learned this lesson.

Using what you have learned, try to improve your work.

If you have not added questions to individual pieces of work, then write your list of questions on the board. Students should select from this list only the questions they think are appropriate to their own work.

If you find you are running out of time, then you could set this task in the next lesson or for homework.

SOLUTIONS

Assessment task: Quadratic Functions

1. a. A matches 3, because it has two positive roots and a positive y-intercept.

   B matches 4, because it has one positive and one negative root.

   C matches 1, because it is the only function with no roots.

   D matches 2 because it is the only function with a maximum value.

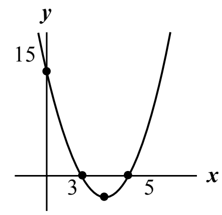
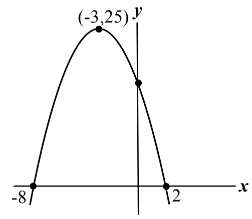
   b. P (6,8); Q (−8,0); R(4,0); S(0, −48).

2. a. $y = (x - 3)^2 - 4$ or $y = x^2 - 6x + 5$

   b. $y = (x - 5)(x - 1)$. The function crosses the x-axis at (5,0) and (1,0).

Collaborative work: *Matching the dominos*

Cards should be placed in this order:

<table>
<tr><td>A
$y = x^2 + 2x - 35$
$y = (x-5)(x+7)$
$y = (x+1)^2 - 36$</td><td>
</td></tr>
<tr><td>H
$y = x^2 - 8x + 15$
$y = (x-3)(x-5)$
$y = (x-4)^2 - 1$</td><td>
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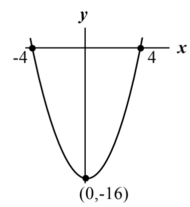
$y = -(x+3)^2 + 25$</td><td>
</td></tr>
<tr><td>F
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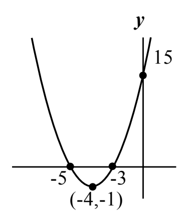
$y = (x-4)(x+4)$
$y = (x-0)^2 - 16$</td><td>
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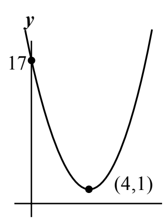
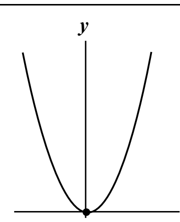
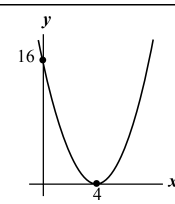
$y = (x+4)^2 - 1$</td><td>
</td></tr>
<tr><td>G
$y = x^2 - 8x + 17$
No roots
$y = (x-4)^2 + 1$</td><td>
</td></tr>
<tr><td>J
$y = x^2$
$y = (x-0)(x-0)$
$y = (x-0)^2 + 0$</td><td>
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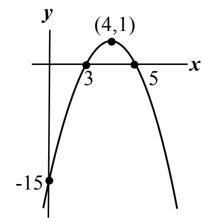
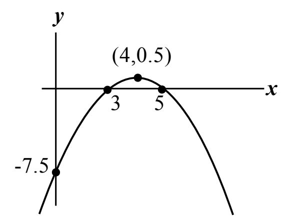
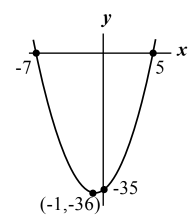
C $y = x^2 - 8x + 16$ $y = (x-4)(x-4)$ $y = (x-4)^2 + 0$	
D $y = -x^2 + 8x - 15$ $y = -(x-3)(x-5)$ $y = -(x-4)^2 + 1$	
I $y = -\frac{1}{2}x^2 + 4x - 7.5$ $y = -\frac{(x-3)(x-5)}{2}$ $y = -\frac{(x-4)^2}{2} + \frac{1}{2}$	

Quadratic Functions

1. Here are 4 equations of quadratic functions and 4 sketches of the graphs of quadratic functions.

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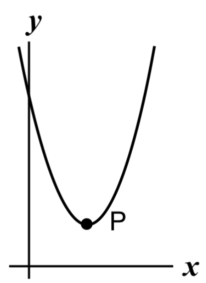
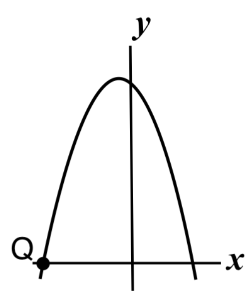
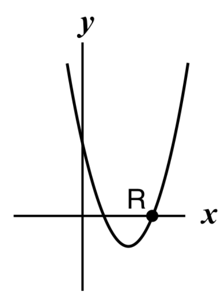
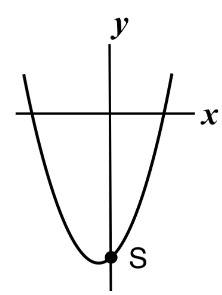
1. 	2. 	3. 	4. 

a. Match the equation to its graph and explain your decision.

Equation A ***matches*** Graph ....., because

Equation B ***matches*** Graph ....., because

Equation C ***matches*** Graph ....., because

Equation D ***matches*** Graph ....., because

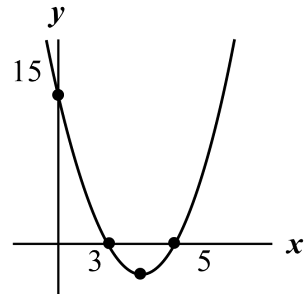
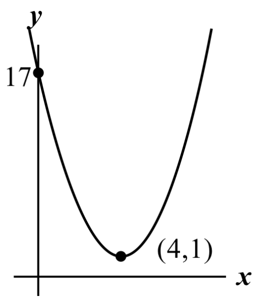
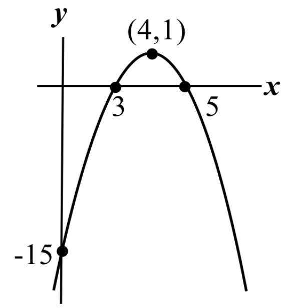
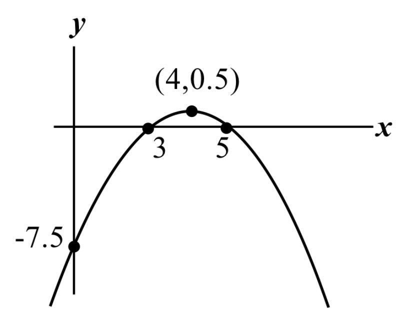
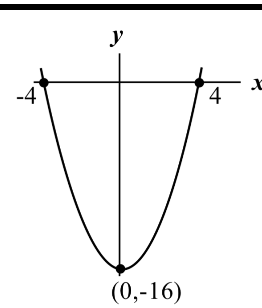
b. Write the co ordinates of the points: P (....,....) Q(....,....) R (....,....) S (....,....)

2. The graph of a quadratic function has a y intercept at (0,5) and a minimum at (3, −4).

a. Write the equation of its curve.

b. Write the coordinates of the root(s) of this quadratic function.

Domino Cards: 1

A $y = x^2 + 2x - 35$ $y =$ .... $y =$	
B $y = x^2 + 8x$.......... $y =$ .... $y = (x+4)^2 - 1$	
C $y = x^2 - 8x$.......... $y = (x-4)(x-4)$ $y =$	
D $y = -x^2 + 8x$.......... $y =$ .... $y = -(x-4)^2 + 1$	
E $y = -x^2 - 6x + 16$ $y = -(x+8)(x-2)$ $y = -(x+3)^2 + 25$	

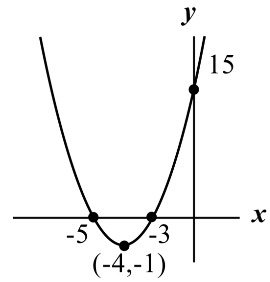
Domino Cards: 2

F

$$y = x^2 \ldots\ldots\ldots\ldots\ldots\ldots$$

$$y = (x-4)(x+4)$$

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

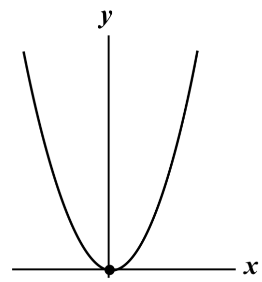


G

$$y = x^2 - 8x \ldots\ldots$$

No roots

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

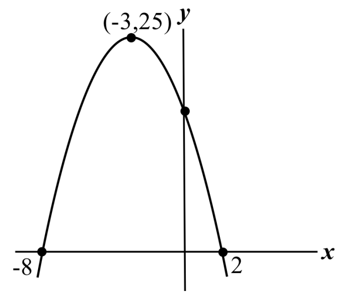


H

$$y = x^2 - 8x + 15$$

$$y = (x-3)(x-5)$$

$$y = (x-4)^2 - 1$$

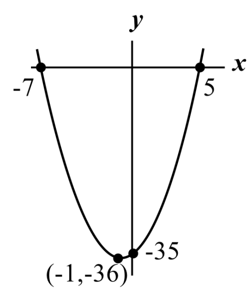


I

$$y = -\frac{1}{2}x^2 + 4x \ldots\ldots\ldots$$

$$y = -\frac{(x-3)(x-5)}{2}$$

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

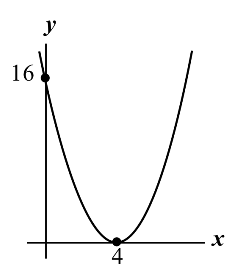


J

$$y = x^2$$

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Matching Dominos

- Take turns at matching pairs of dominos that you think belong together.
- Each time you do this, explain your thinking clearly and carefully to your partner.
- It is important that you both understand the matches. If you don't agree or understand, ask your partner to explain their reasoning. You are both responsible for each other's learning.
- On some cards an equation or part of an equation is missing. Do not worry about this, as you can carry out this task without this information.

Sharing Work

- One student from each group is to visit another group's poster.

- If you are staying at your desk, be ready to explain the reasons for your group's matches.

- If you are visiting another group:
  - Write your card matches on a piece of paper.

  - Go to another group's desk and check to see which matches are different from your own.

  - If there are differences, ask for an explanation. If you still don't agree, explain your own thinking.

  - When you return to your own desk, you need to consider as a pair whether to make any changes to your own work.

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
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at the
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based on their observation of trials in US classrooms
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