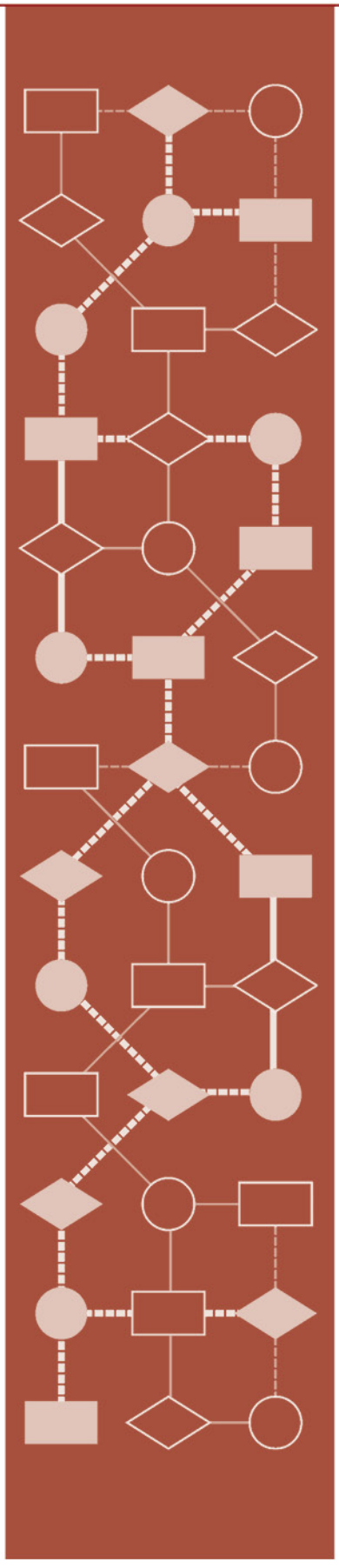
PROBLEM SOLVING



Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Sharing Costs:
Travelling to School

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Sharing Costs: *Travelling to School*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to solve a real-world modeling problem. There are several correct approaches to the problem, including some that involve proportional relationships.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

1 Make sense of problems and persevere in solving them.

2. Reason abstractly and quantitatively.

4. Model with mathematics.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics:*

6.RP Understand ratio concepts and use ratio reasoning to solve problems.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You then review their work and create questions for them to answer in order to improve their solutions.
- At the start of the lesson students work alone, answering your questions about the same problem. They are then grouped and engage in a collaborative discussion of the same task.
- In the same small groups, students are given sample solutions to comment on and evaluate.
- In a whole-class discussion, students explain and compare the alternative solution strategies they have seen and used.
- Finally, students reflect on their work.

MATERIALS REQUIRED

- Each student will need two copies of the assessment task *Sharing Gasoline Costs*, the review questionnaire *How Did You Work?*, a mini-whiteboard, a pen, and an eraser.
- Each small group of students will need copies of all three *Sample Student Responses*, a large sheet of paper for making a poster, and a marker pen.
- There are some projector resources to support whole-class discussions.

Graph paper should be available for students who require it.

TIME NEEDED

20 minutes before the lesson, an 80-minute lesson and 20 minutes in a follow-up lesson (or for homework). Timings given are only approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

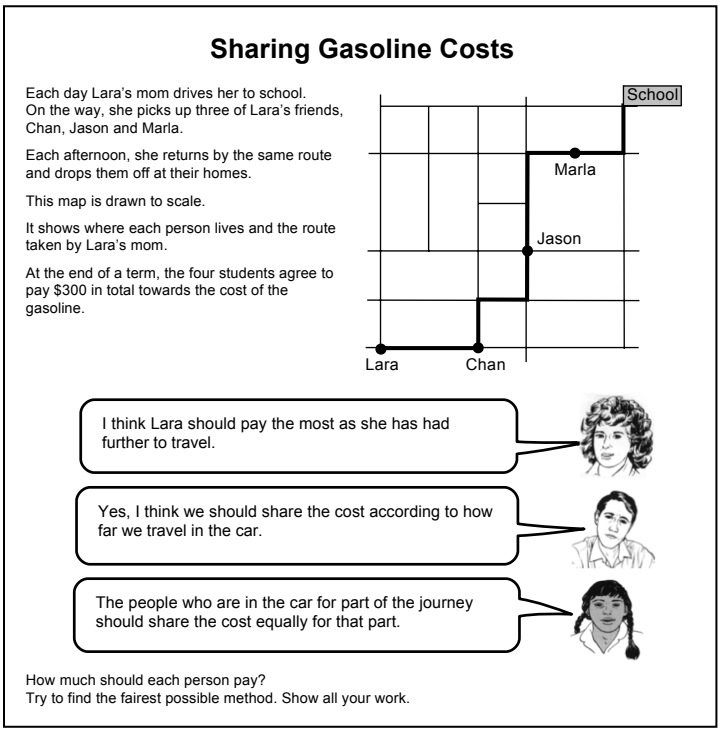
Assessment task: *Sharing Gasoline Costs* (20 minutes)

Give each student a copy of the assessment task *Sharing Gasoline Costs*.

> *Read through the sheet and try to answer the question as carefully as you can.*
>
> *Try to present your work in an organized and clear manner, so everyone can understand it.*

It is important that, as far as possible, students are allowed to complete the task without your assistance.

Students who sit together often produce similar answers so that when they come to compare their work, they have little to discuss. For this reason we suggest that when students do the task individually you ask them to move to different seats. Then at the beginning of the formative assessment lesson allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.



Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. Research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- Write one or two questions on each student's work, or
- Give each student a printed version of your list of questions, and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students, and write these on the board when you return the work to the students.

Common issues	Suggested questions and prompts
Student takes an unproductive approach	• Read through the problem again. What are you trying to figure out? Is your method helping you to get there? • Are there any other ways of approaching the problem that might be productive? • The comments in the dialogue boxes may help you get started.
Student gives answers without considering a 'fair' mathematical approach For example: Student just gives numbers that total $300.	• How can you justify your answers mathematically? • How many blocks does Marla travel in the car? How many blocks does Jason travel in the car? What could you say about the amount Jason should pay compared to the amount Marla should pay?
Student states that each person should contribute $75 to the cost of gasoline	• Is it fair for Marla to pay the same amount as Lara? Why? Why not?
Student uses an inefficient method For example: The student figures out the total distances travelled for the whole term. Or: The student figures out the distances travelled to and from school.	• Your calculations use the distances travelled for the whole term. Can you now think of a quicker way to arrive at the same answers? • Do you need to consider both the outward and return journeys? Why? Why not? • Does taking into account the return journey affect how the cost of the gasoline is shared out?
Student work lacks explanation For example: The student presents their work as an algebraic equation without any explanation. Or: The student does not explain why they have not taken into account the return journey.	• Would someone unfamiliar with this type of work understand your solution? • Students do have to pay for each return journey. Explain why your work does not take this into account.
Student makes a technical error For example: The student makes an arithmetical error.	• Check your calculations.
Student correctly answers all the questions. Student needs an extension task.	• Now think of a different way the cost of the gasoline could be shared fairly between the students.

SUGGESTED LESSON OUTLINE

Individual work (10 minutes)

Return the assessment task to the students. Give each student a mini-whiteboard, a pen, and an eraser.

Begin the lesson by briefly reintroducing the problem.

If you did not add questions to individual pieces of work, write your list of questions on the board. Ask students to select questions appropriate to their own work, and spend a few minutes answering them.

Recall what we were looking at in a previous lesson. What was the task about?

Today we are going to work together to try to improve your initial attempts at this task.

I have had a look at your work, and I have some questions I would like you to think about.

On your own, carefully read through the questions I have written. I would like you to use the questions to help you to think about ways of improving your own work.

Use your mini-whiteboards to make a note of anything you think will help to improve your work.

Collaborative small-group work (20 minutes)

Organize the class into small groups of two or three students.

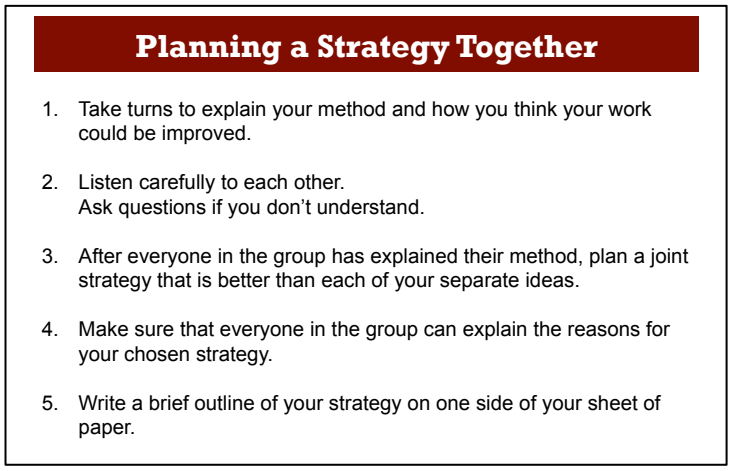
Give each group a large piece of paper and a marker pen. On request, give students graph paper.

Deciding on a Strategy

Ask students to share their ideas about the task and plan a joint solution.

I want you to share your work with your group.

Slide P-1 of the projector resource, *Planning a Strategy Together*, explains the process.



Once students have evaluated the relative merits of each approach ask them to write their strategy on the large piece of paper.

Implementing the Strategy

Students are now to turn their large sheet of paper over and write their joint solution clearly in the form of a poster.

While students work together in small groups you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches to the task

Listen and watch students carefully. Note different approaches to the task and the assumptions that students make. Do students work systematically? Do students use proportional reasoning? In their calculations, do students use the distances travelled for a term, a week, a day, or a single journey? Do students use ratios? Do students use algebra? In particular, note any common mistakes. You can then use this information to focus a whole-class discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking. In particular focus on the strategies rather than the solution. Encourage students to justify their statements.

You may want to use the questions in the *Common issues* table to support your own questioning. If the whole class is struggling on the same issue, you could write one or two relevant questions on the board and hold a brief whole-class discussion. You could also give any struggling students one of the sample responses. Encourage students to write down their method on the poster, as it will help everyone in the group to understand and evaluate it.

Whole-class discussion (10 minutes)

You may want to hold a brief whole-class discussion. Have students solved the problem using a variety of methods? Or have you noticed some interesting ways of working or some incorrect methods; if so, you may want to focus the discussion on these. Equally, if you have noticed different groups using similar strategies but making different assumptions you may want to compare solutions.

Collaborative analysis of *Sample Student Responses* (20 minutes)

Distribute copies of *Sample Student Responses* to each group of students. This task gives students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy.

There may not be time, and it is not essential, for all groups to look at all three sample responses. If this is the case, be selective about what you hand out. For example, groups that have successfully completed the task using one method will benefit from looking at a different approach. Other groups that have struggled with a particular approach may benefit from seeing a student version of the same strategy.

> *In your groups you are now going to look at some student work on the task. Notice in what ways this work is similar to yours and in which ways it is different.*
>
> *There are some questions for you to answer as you look at the work. You may want to add notes to the work to make it easier to follow.*

Slide P-2 of the projector resource, *Evaluating Sample Student Responses* describes how students should work together:

Evaluating Sample Student Responses

1. Take turns to work through each student's solution.
   Answer the questions individually on your mini-whiteboards.
2. Explain your answers to the rest of the group.
3. Listen carefully to explanations.
   Ask questions if you don't understand.
4. When you all agree with the explanations, write the answers below the sample work.

   Make sure the student who writes the answers is not the student who explained them.

Encourage students to focus on evaluating the strategies and math in the student work, not on superficial features, such as whether or not the student has neat handwriting.

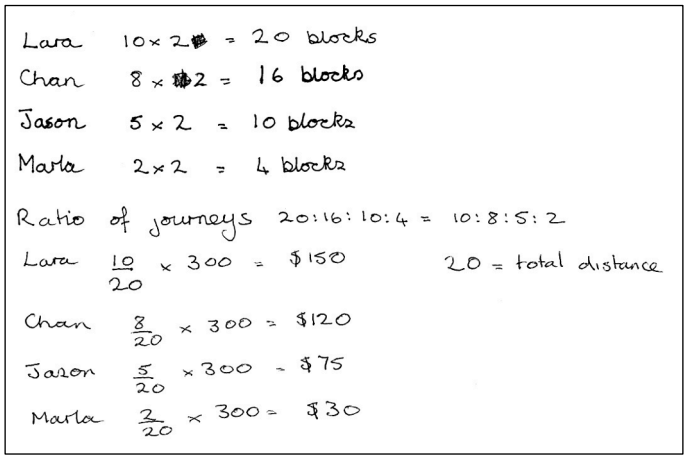
During the small group work, support the students as in the first collaborative activity. Also, check to see which of the explanations students find more difficult to understand. Note similarities and differences between the sample approaches and those the students took in the collaborative group work.

Adam used ratios. His strategy is to share out the cost in proportion to the distance each student travels.

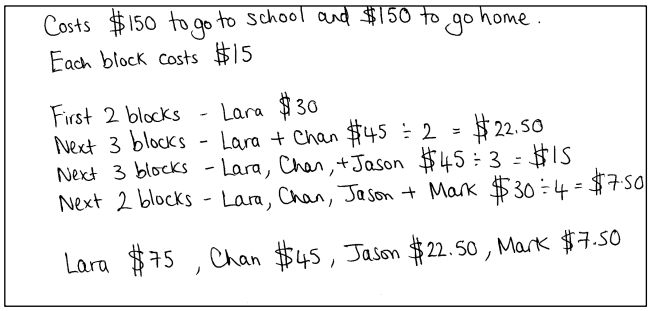
He has figured out the distance travelled to and from school for each student and written these distances as ratios. Adam has simplified the ratios.

Adam makes a mistake when figuring out the costs for each student. He should have figured out the total distance traveled (50 blocks) rather than the maximum distance traveled by one student (20 blocks).

What does each fraction represent?
Is this the correct fraction to use when figuring out each student's costs?
Why/why not?



Kimberley has correctly figured out the cost of each block. She has then split this cost between the students in the car at different parts of the journey. She has only figured out the cost of getting to school. All Kimberley's answers should be doubled. Her explanations are not clear.


Costs $150 to go to school and $150 to go home.
Each block costs $15

First 2 blocks - Lara $30
Next 3 blocks - Lara + Chan $45 ÷ 2 = $22.50
Next 3 blocks - Lara, Chan, + Jason $45 ÷ 3 = $15
Next 2 blocks - Lara, Chan, Jason + Mark $30 ÷ 4 = $7.50

Lara $75, Chan $45, Jason $22.50, Mark $7.50

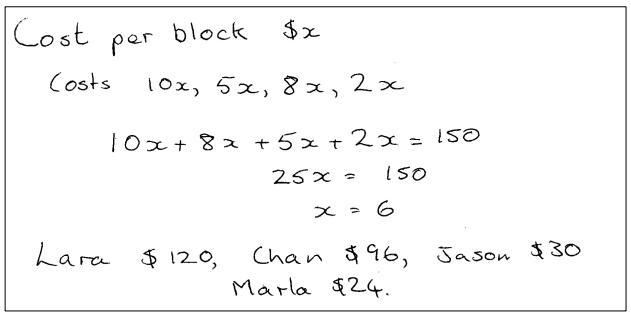
How could Kimberley improve the presentation of her work? [Writing the amounts in a table.]

Has Kimberley used proportional reasoning? [She has tried to allocate costs in proportion to the number in the car at particular times]

Students may think this method is the fairest one as the costs take account of shared lifts.

Donna uses algebra to figure out the costs. Donna's method shares out the cost in proportion to the distance each student travels to school.

Her explanations are not clear and her answer for Jason is incorrect as she has failed to double the amount.


Cost per block $x

Costs $10x, 5x, 8x, 2x$

$10x + 8x + 5x + 2x = 150$

$25x = 150$

$x = 6$

Lara $120, Chan $96, Jason $30
Marla $24.

Has Donna used proportional reasoning?

Whole-class discussion: comparing different approaches (20 minutes)

Hold a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on parts of the task students found difficult. Ask the students to compare the different solution methods.

Which approach did you think was the fairest and most effective? Why?

Which approach did you find most difficult to understand? Why?

How could the student improve his/her answer?

Did anyone come up with a method different from these?

Try to focus the discussion on any common misconceptions you noticed in the collaborative work. You may want to draw on the questions in the *Common issues* table to support your own questioning. Try to resist simply explaining the mistakes students have made.

To support the discussion, you may want to project the sample work using Slides P-3, P-4, and P-5 of the projector resource.

Review solutions to *Sharing Gasoline* (20 minutes)

Give out the sheet *How Did You Work?* and ask students to complete this questionnaire. The questionnaire should help students review their progress.

If you have time you may also want to ask your students to read through their original solutions and using what they have learned, attempt the task again, perhaps using a different method. In this case, give each student a blank copy of the assessment *Sharing Gasoline*.

Some teachers give this task as a homework task.

SOLUTIONS

Assessment task: *Sharing Gasoline Costs*

Two different answers to this task are presented below. The first shares out the cost in proportion to the distance each student traveled, whereas the second takes into account the number of people in the car for each part of the journey. Both answers are valid.

N.B. Some students may decide that Lara should not pay as her mom is driving and base their calculations on the three friends Chan, Jason and Marla. Also, we make no mention that Lara's mom (assuming she does not stay at school all day) might actually make **four** journeys per day, and for two of these the car contains no passengers.

Adam's method (improved)

Lara's journey to school: 10 blocks
Chan's journey to school: 8 blocks
Jason's journey to school: 5 blocks
Marla's journey to school: 2 blocks

Ratio of distances travelled: 10 : 8 : 5 : 2.

Total parts: 25.

Total cost of gasoline: $75 \times 4 = \$300$. This cost is split in the ratio of the distances travelled.

$$\text{Lara}: \frac{10}{25} \times 300 = \$120$$

$$\text{Chan}: \frac{8}{25} \times 300 = \$96$$

$$\text{Jason}: \frac{5}{25} \times 300 = \$60$$

$$\text{Marla}: \frac{2}{25} \times 300 = \$24$$

This method shares out the cost in proportion to the distance each student travels.

Kimberley's method (improved)

Total cost of gasoline: $300.

Total distance Lara's mom drives each day: 20 blocks.

Cost of gasoline per block travelled: $300 ÷ 20 = $15.

		Lara	Chan	Jason	Marla
First 2 blocks	$30	$30			
Next 3 blocks	$45	$22.50	$22.50		
Next 3 blocks	$45	$15	$15	$15	
Last 2 blocks	$30	$7.50	$7.50	$7.50	$7.50
Total costs for journey to school		$75	$45	$22.50	$7.50
Total costs for journeys to and from school		$150	$90	$45	$15

Donna's method (improved)

Total cost of gasoline for the journey to school = \$150.

Cost of gasoline per block travelled to school: \$$x$

Lara's journey to school:	10 blocks.	Cost for Lara to travel to school:	$10x$
Chan's journey to school:	8 blocks.	Cost for Chan to travel to school:	$8x$
Jason's journey to school:	5 blocks.	Cost for Jason to travel to school:	$5x$
Marla's journey to school:	2 blocks.	Cost for Marla to travel to school:	$2x$

$$10x + 8x + 5x + 2x = 150$$

$$25x = 150$$

$$x = 6$$

Cost for Lara to travel to and from school: $10 \times 6 \times 2 = \$120$

Cost for Chan to travel to and from school: $8 \times 6 \times 2 = \$96$

Cost for Jason to travel to and from school: $5 \times 6 \times 2 = \$60$

Cost for Marla to travel to and from school: $2 \times 6 \times 2 = \$24$.

This method shares out the cost in proportion to the distance each student travels.

Sharing Gasoline Costs

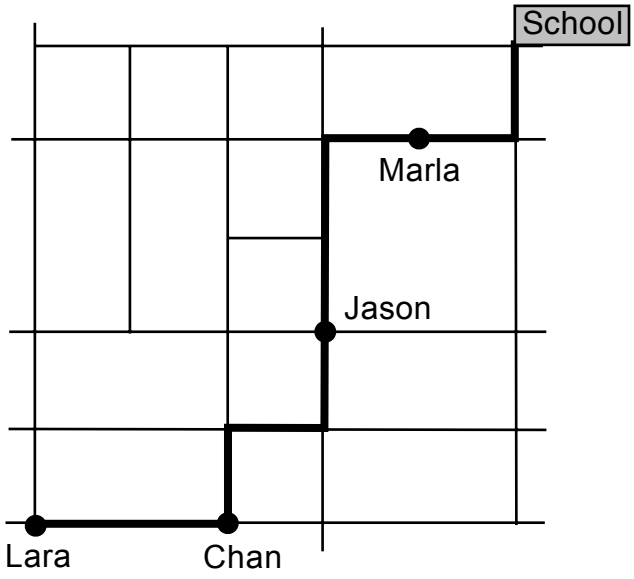
Each day Lara's mom drives her to school.
On the way, she picks up three of Lara's friends, Chan, Jason and Marla.

Each afternoon, she returns by the same route and drops them off at their homes.

This map is drawn to scale.

It shows where each person lives and the route taken by Lara's mom.

At the end of a term, the four students agree to pay $300 in total towards the cost of the gasoline.



I think Lara should pay the most as she has had further to travel.

Yes, I think we should share the cost according to how far we travel in the car.

The people who are in the car for part of the journey should share the cost equally for that part.

How much should each person pay?
Try to find the fairest possible method. Show all your work.

Sample Student Response: Adam

Lara $10 \times 2 = 20$ blocks

Chan $8 \times 2 = 16$ blocks

Jason $5 \times 2 = 10$ blocks

Marla $2 \times 2 = 4$ blocks

Ratio of journeys $20:16:10:4 = 10:8:5:2$

Lara $\frac{10}{20} \times 300 = \$150$ 20 = total distance

Chan $\frac{8}{20} \times 300 = \$120$

Jason $\frac{5}{20} \times 300 = \$75$

Marla $\frac{2}{20} \times 300 = \$30$

What math does Adam do correctly? Explain.

Has Adam made any mistakes? Explain.

Sample Student Response: Kimberley

Costs $150 to go to school and $150 to go home.
Each block costs $15

First 2 blocks - Lara $30
Next 3 blocks - Lara + Chan $45 ÷ 2 = $22.50
Next 3 blocks - Lara, Chan, + Jason $45 ÷ 3 = $15
Next 2 blocks - Lara, Chan, Jason + Mark $30 ÷ 4 = $7.50

Lara $75, Chan $45, Jason $22.50, Mark $7.50

Why does each block cost $15?

Does Kimberley take into account the return journey? Does this affect the answer? Explain.

How has Kimberley split the cost of the gasoline?

Sample Student Response: Donna

Cost per block $x

Costs 10x, 5x, 8x, 2x

$$10x + 8x + 5x + 2x = 150$$
$$25x = 150$$
$$x = 6$$

Lara $120, Chan $96, Jason $30
Marla $24.

Does Donna take into account the return journey? Does this affect the answer? Explain.

How has Donna split the cost of the gasoline?

How could Donna improve her work?

How Did You Work?

Complete the sentences and mark the boxes that apply to your work.

1. The method I used to complete the task on my own was:

   The method we used in our group was:

   The method I preferred was:

   because:

2. Our solution is similar to one of the sample responses ☐ **OR** Our solution is different from **all** the sample responses ☐

   Our solution is similar to _____ *add name of the sample response*

   Our solution is different from **all** the sample responses because:

   I prefer **our solution / the sample response** because:

3. We checked our work by:

Planning a Strategy Together

1. Take turns to explain your method and how you think your work could be improved.

2. Listen carefully to each other.
   Ask questions if you don't understand.

3. After everyone in the group has explained their method, plan a joint strategy that is better than each of your separate ideas.

4. Make sure that everyone in the group can explain the reasons for your chosen strategy.

5. Write a brief outline of your strategy on one side of your sheet of paper.

Evaluating Sample Student Responses

1. Take turns to work through each student's solution.
   Answer the questions individually on your mini-whiteboards.

2. Explain your answers to the rest of the group.

3. Listen carefully to explanations.

   Ask questions if you don't understand.

4. When you all agree with the explanations, write the answers below the sample work.

   Make sure the student who writes the answers is not the student who explained them.

Sample Student Response: Adam

Lara 10×2 = 20 blocks

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Ratio of journeys 20:16:10:4 = 10:8:5:2

Lara $\frac{10}{20} \times 300$ = \$150 20 = total distance

Chan $\frac{8}{20} \times 300$ = \$120

Jason $\frac{5}{20} \times 300$ = \$75

Marla $\frac{2}{20} \times 300$ = \$30

Sample Student Response: Kimberley

Costs $150 to go to school and $150 to go home.
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Sample Student Response: Donna

Cost per block $x

Costs 10x, 5x, 8x, 2x

$$10x + 8x + 5x + 2x = 150$$

$$25x = 150$$

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Lara $120, Chan $96, Jason $30

Marla $24.

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
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It was refined on the basis of reports from teams of observers led by
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along with comments from teachers and other users.

This project was conceived and directed for
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and based at the University of California, Berkeley

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