CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Modeling Conditional Probabilities: 2

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Modeling Conditional Probabilities 2

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students understand conditional probability, and, in particular, to help you identify and assist students who have the following difficulties:

- Representing events as a subset of a sample space using tables and tree diagrams.
- Understanding when conditional probabilities are equal for particular and general situations.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*

S-CP: Understand independence and conditional probability and use them to interpret data.
S-MD: Calculate expected values and use them to solve problems.
F-BF: Build a function that models a relationship between two quantities.
A-SSE: Write expressions in equivalent forms to solve problems.

This lesson also relates to the following Standards for Mathematical Practice in the Common Core State Standards for Mathematics

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
4. Model with mathematics.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understandings and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- During the lesson, students first work collaboratively on a related task. They have an opportunity to extend and generalize this work. The lesson ends with a whole-class discussion.
- In a subsequent lesson, students revise their individual solutions to the assessment task.

MATERIALS REQUIRED

- Each student will need two copies of the assessment task, *A Fair Game*, a mini-whiteboard, a pen, and an eraser.
- Each pair of students will need a copy of the sheet *Make It Fair* cut up into cards, a felt-tipped pen, and a large sheet of paper for making a poster.
- You will need several copies of the extension material, *Make It Fair: Extension Task*, a bag, and some balls.
- There are some projector resources to support whole class discussion.

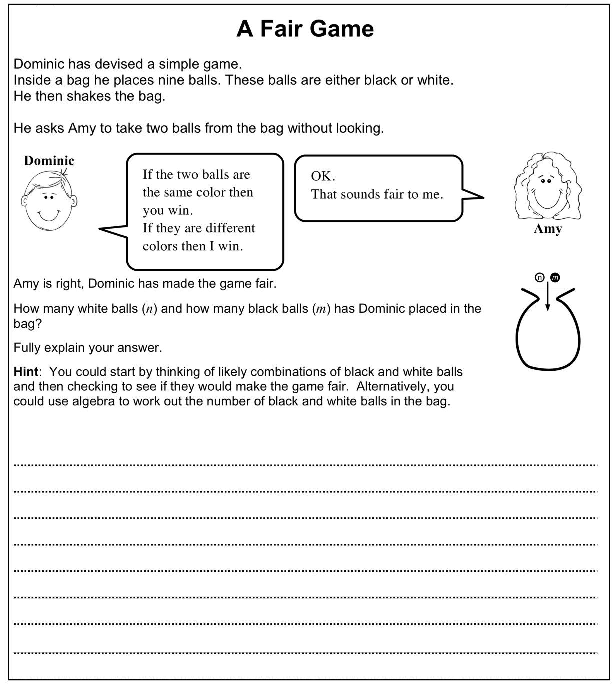
TIME NEEDED

Approximately 20 minutes before the lesson, a 1-hour lesson, and 15 minutes in a follow-up lesson. All timings are approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *A Fair Game* task (20 minutes)

Set this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work, and to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the next lesson.



A Fair Game

Dominic has devised a simple game.
Inside a bag he places nine balls. These balls are either black or white.
He then shakes the bag.

He asks Amy to take two balls from the bag without looking.

Amy is right, Dominic has made the game fair.

How many white balls (n) and how many black balls (m) has Dominic placed in the bag?

Fully explain your answer.

Hint: You could start by thinking of likely combinations of black and white balls and then checking to see if they would make the game fair. Alternatively, you could use algebra to work out the number of black and white balls in the bag.

Give each student a copy of the assessment task, *A Fair Game*.

Make sure the class understands the rules of the game and what is meant by a 'fair game' by demonstrating, using a bag and some balls.

> *Read through the questions and try to answer them as carefully as you can.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We recommend that you write a selection of questions on each student's work. If you do not have time, select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson.

Common issues	Suggested questions and prompts
Student has trouble getting started For example: The student produces no work.	• Imagine there are only two balls in the bag, both black. List all the possible outcomes. Would the game be fair? What if there were one white and one black ball? Two white balls? • Now consider the game with three balls in the bag.
Student does not define an event consistently	• Describe how Amy picks out the two balls. Does it make any difference whether she picks two at the same time, or one after another?
Student does not choose a suitable representation	• Can you think of a suitable diagram that will show all the possibilities? • Can you use a sample space diagram?
Student does not recognize dependent probabilities For example: The probability that the second ball is black is assumed to be independent of the choice of the first ball. So P(both balls black) is assumed to be 0.5×0.5.	• Imagine each ball is taken out of the bag one at a time. When one ball is taken out of the bag, how many remain in the bag? How does this affect the math?
Student selects the same ball twice in their table of possible outcomes For example: The student assumes that there are $3 \times 3 = 9$ ways of obtaining two black balls.	• Is it possible to select the same ball twice?
Presentation of the work is incomplete For example: The student does not fully label the tree diagram or the sample space diagram.	• Would someone unfamiliar with this type of work understand the math?
Student is unable to form an algebraic equation	• Suppose n is the number of white balls. In terms of n, how many black balls are in the bag? • Look at your sample space diagram. How can you quickly count the number of ways Amy can win? Express this algebraically. How can you quickly count the number of ways Amy can lose? Express this algebraically. • What makes the game fair? Show this as an algebraic equation.
Student makes a technical algebraic mistake	• Check your solution.
Student correctly draws sample space diagram	• Now use the variable n and your sample space diagram to figure out an algebraic method. • To make the game fair, what values in your sample space must be equal? How can you show this algebraically?
Student correctly draws tree diagram	• Draw a sample space to check your solution.
Student correctly answers all the questions	• Suppose there are eight balls in the bag, is it possible to make the game fair? Why/Why not?

SUGGESTED LESSON OUTLINE

Re-introduce the problem (5 minutes)

This lesson assumes students are familiar with sample space diagrams.

Introduce the lesson:

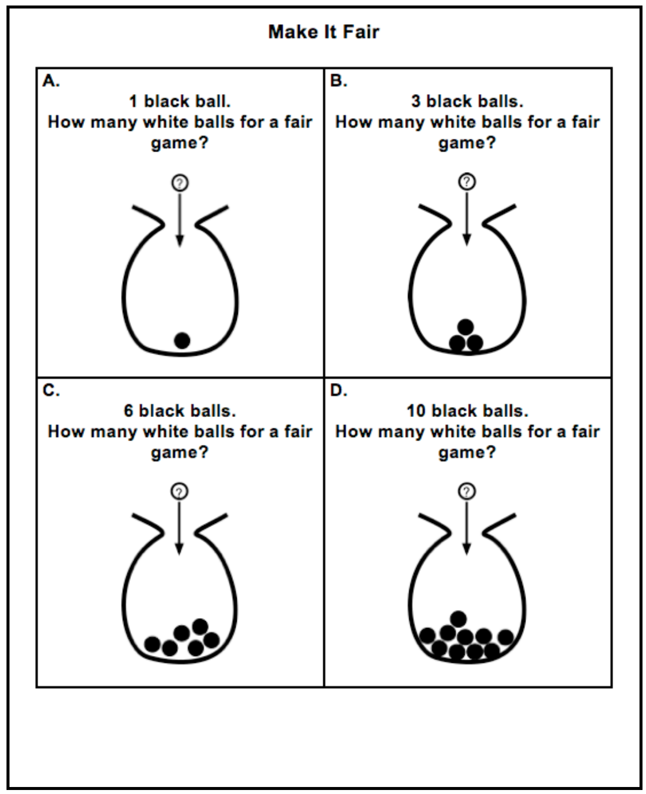
Recall what you were working on in the assessment. What was the task?

Explain that in this lesson the students will be working collaboratively on a similar task to the one in the assessment.

Remind students of the rules, and what is meant by a 'fair game', by demonstrating the game using a bag and some balls.

Slide P-1 of the projector resource, *Make it Fair*, outlines the rules of the game.

Collaborative work: *Make it Fair* (30 minutes)



Organize the students into pairs.

Give each pair one of the cards A - D from *Make It Fair,* a large sheet of paper, and a felt-tipped pen for making a poster. Use your judgment from the assessment to decide which student gets which card.

- If you think that students will struggle, give them **one** of cards A or B. These have just one or three black balls in the bag.
- If you think that some will find the task more straightforward, give them **one** of cards C or D. These have six or ten black balls in the bag.

Try to give all of four cards to your class.

[Note: We have chosen the numbers 1, 3, 6, 10 for number of black balls because these all give solutions (3, 6, and 10 give two solutions). Not every number of black balls gives a solution. Do not tell students this!]

Introduce the task carefully:

You are now going to continue with the same game you worked on in the assessment, but this time there are some black balls already in the bag.

Your task is to determine how many white balls should be added to the bag to make the game fair.

There may be more than one answer for each card.

Try a number of white balls, and if it doesn't work out to be a fair game, think about why it doesn't. Then try to find a different solution.

It is important that you both understand and agree with the solution. If you don't, explain why. You are both responsible for each other's learning.

Prepare a presentation of your problem and your work on the poster.

Slide P-2 of the projector resource, *Working Together*, summarizes these instructions.

Encourage students not to rush, but spend time fully justifying their answers to one card.

While students work in pairs you have two tasks: to note different student approaches to the task, and to support student problem solving.

Note different student approaches to the task

There are many ways students may organize possible outcomes, including a network, a graph, a table, a tree diagram, or a sample space diagram. If they choose not to draw a sample space diagram, encourage them to try the sample space diagram as well, and to decide for themselves which approach is easier (see below). Students may get confused about what the 'event' is: Amy picking a pair of balls, or the serial events, picking one ball, then another.

Support student problem solving

Ask questions that help students clarify their thinking. In particular, focus on the strategies rather than the solution. You may want to use some of the questions from the table, *Common issues*.

How does your diagram help you answer the question?

Now draw a sample space diagram and decide which you prefer.

Encourage students to work out a quick way of finding a solution.

If students are succeeding, then encourage them to generalize their reasoning.

What is a quick way of working out the number of ways for Amy to win/lose for this sample space?

Instead of drawing a sample space, how can you use algebra to work out the number of white balls required to make the game fair?

If there are n white balls and this number of black balls, how can you work out n?
To make the game fair, what values in your sample space diagram must be equal?
Now create two algebraic expressions for these two values.

If students are having difficulty with the task, encourage them to draw a sample space in the form of an organized table. For example, if there are six black balls and two white balls in the bag, the student might draw the following diagram. Here, a check mark indicates a matching pair (a win for Amy) and a cross indicates a pair that doesn't match (Amy loses).

		First selection							
		B_1	B_2	B_3	B_4	B_5	B_6	W_1	W_2
Second selection	B_1	-	✓	✓	✓	✓	✓	x	x
	B_2	✓	-	✓	✓	✓	✓	x	x
	B_3	✓	✓	-	✓	✓	✓	x	x
	B_4	✓	✓	✓	-	✓	✓	x	x
	B_5	✓	✓	✓	✓	-	✓	x	x
	B_6	✓	✓	✓	✓	✓	-	x	x
	W_1	x	x	x	x	x	x	-	✓
	W_2	x	x	x	x	x	x	✓	-

How many ways of winning are there? *[32]*
What is a quick way of counting all the possible wins for Amy? *[E.g. (6×6−6)+(2×2−2) =32.]*

How many ways of Amy losing are there? *[24]*
What is a quick way of counting all the crosses? *[E.g. 2×6×2 = 24.]*

Is the game fair?

If you added another white ball, how would the diagram change?
How would your calculations change?

Is the game fair now?

So, for six black and n white balls:

- The number of winning combinations = $(6^2 - 6) + n^2 - n = 30 + n^2 - n$.
- The number of losing combinations = $2 \times 6n = 12n$.

They might then equate these and solve the resulting quadratic to obtain:

$$30 + n^2 - n = 12n$$
$$n^2 - 13n + 30 = 0$$
$$(n-3)(n-10) = 0$$
$$n = 3 \text{ or } n = 10$$

This shows that, when the number of black balls is 6, the game is fair only when the number of white balls is 3 or 10.

If students succeed in working out the number of white balls algebraically, then ask them to check their method with a different card.

Extension task

If students progress quickly ask them to try one of the extension cards. Cards E and F have no solutions. Cards G and H each have two solutions, but the work on these is complex so it should encourage students to consider structure and move towards using algebra. Card I is the most difficult.

Whole-class discussion (15 minutes)

Invite pairs of pupils to show their posters and describe their reasoning. Then ask other pairs who have worked on the same card to describe their reasoning.

Ask the remaining students to say which reasoning they found most clear and convincing, and why.

The important part here is to share reasoning clearly and carefully, not to arrive at a particular set of results. If your class stops here, you will have achieved a great deal.

After the presentations students may benefit from spending some time reviewing their poster. Some classes may like to go on and generalize their results (these include the extension cards):

Number of black balls	**Possible number(s) of white balls for a fair game**
1	3
2	No solutions
3	1 or 6
4	No solutions
6	3 or 10
10	6 or 15
15	10 or 21
21	15 or 28

Can you see a pattern? [The numbers 1, 3, 6, 10, 15, 21, 28 are all ‘triangle numbers’.]

What do you notice about the total number of balls in the bag when the game is fair? [It is always a square number.]

What do you notice about the difference between the number of black balls and the number of white balls? [It is the positive square root of the square number.]

Suppose 15 black balls are placed in the bag. How many white balls will you need to add to make the game fair? Can you check your answer? [10 or 21]

Follow-up lesson: *A Fair Game* (15 minutes)

Return the original assessment *A Fair Game* to the students, and a second, blank copy of the task.

Look at your original responses and think about what you have learned this lesson.

Using what you have learned, try to improve your work.

If you have not added questions to individual pieces of work, write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work. However, some students may struggle to identify which questions they should consider from this list. If this is the case, it may be helpful to give students a printed version of the list of questions so that you can highlight the ones that you want them to focus on.

SOLUTIONS

Assessment: *A Fair Game*

In the bag there are either 3 black ball and 6 white balls or 6 black balls and 3 white balls.

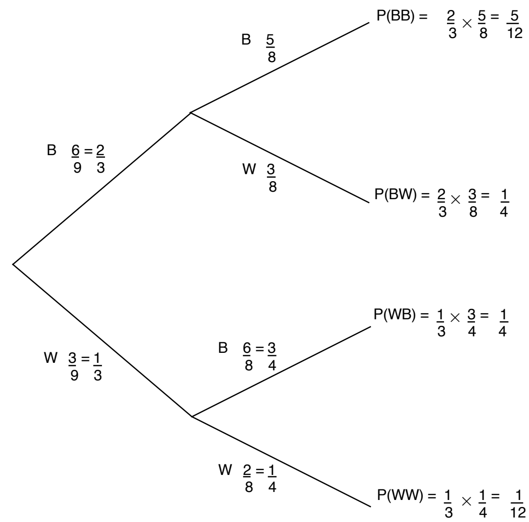
The solution can be worked out by ‘guess and check’. Guess combinations, and then check by drawing a tree diagram or a sample space. Alternatively, students can work out an algebraic solution.

Guess and check:

The correct sample space:

		First selection								
		B_1	B_2	B_3	B_4	B_5	B_6	W_1	W_2	W_3
Second selection	B_1	-	✓	✓	✓	✓	✓	x	x	x
	B_2	✓	-	✓	✓	✓	✓	x	x	x
	B_3	✓	✓	-	✓	✓	✓	x	x	x
	B_4	✓	✓	✓	-	✓	✓	x	x	x
	B_5	✓	✓	✓	✓	-	✓	x	x	x
	B_6	✓	✓	✓	✓	✓	-	x	x	x
	W_1	x	x	x	x	x	x	-	✓	✓
	W_2	x	x	x	x	x	x	✓	-	✓
	W_3	x	x	x	x	x	x	✓	✓	-

The correct tree diagram:



Algebraic solution 1:

If there are n black balls and $9 - n$ white balls:

Number of ways Amy could win:

$= $ number of ways of selecting two balls the same

$= (n^2 - n) + ((9 - n)^2 - (9 - n)) = n^2 - n + 81 - 18n + n^2 - 9 + n = 2n^2 - 18n + 72$

Number of ways Dominic could win:

$= $ number of ways of selecting a black ball + number of ways of selecting a white ball

$= n(9 - n) + n(9 - n) = 18n - 2n^2$

The game is fair when:

$2n^2 - 18n + 72 = 18n - 2n^2$

$4n^2 - 36n + 72 = 0$

$n^2 - 9n + 18 = 0$

$(n - 3)(n - 6) = 0$

$n = 3$ or $n = 6$.

Algebraic solution 2:

If there are n black balls and m white balls:

Number of ways Amy could win:

= number of ways of picking two balls the same

$= (n^2 - n) + (m^2 - m)$

Number of ways Dominic could win:

= number of ways of selecting a black ball + number of ways of selecting a white ball

$= mn + mn = 2mn$;

The game is fair when:

$(n^2 - n) + (m^2 - m) = 2mn$

$(n - m)^2 = n + m$

The total number of balls in the bag is a square number (9).

Therefore $3 = n - m$

So $m = 6$ and $n = 3$ or $m = 3$ and $n = 6$.

Collaborative work: *Make it Fair*

The following combinations of balls make the game fair:

A. 1 black ball and 3 black balls.

B. 3 black balls and 1 or 6 white balls.

C. 6 black balls and 10 or 3 white balls.

D. 10 black balls and 6 or 15 white balls.

E. It is not possible to make the game fair.

F. It is not possible to make the game fair.

G. 15 black balls, 10 or 21 white balls.

H. 21 black balls, 28 or 15 white balls.

I. m black balls - this general case is considered below.

Suppose there are m black balls and n white balls. Here is an algebraic version of the sample space:

		1st selection	
		m black balls B_1 B_2 B_3 ……… B_M	n white balls W_1 W_2 W_3 ……… W_N
2nd selection	m black balls	B_1 B_2 B_3 …. …. B_M	Number of possible combinations for selecting two black balls: $m^2 - m$	Number of possible combinations for selecting two different colored balls: mn
	n white balls	W_1 W_2 W_3 …. …. W_N	Number of possible combinations for selecting two different colored balls: mn	Number of possible combinations for selecting two white balls: $n^2 - n$

For the game to be fair:

Number of pairs with same color = number of pairs with different colors.

$$m^2 - m + n^2 - n = 2mn$$

$$n^2 - 2mn + m^2 = n + m$$

$$(n - m)^2 = m + n \qquad (1)$$

This shows two interesting facts:

The total number of balls must be a perfect square.

The difference between the number of black and the white balls is the square root of this number.

Furthermore, suppose we let $n - m = k,$ then from (1):

$$k^2 = 2m + k \qquad\qquad k^2 = 2n - k$$

$$m = \frac{k(k-1))}{2} \qquad\qquad n = \frac{k(k+1))}{2}$$

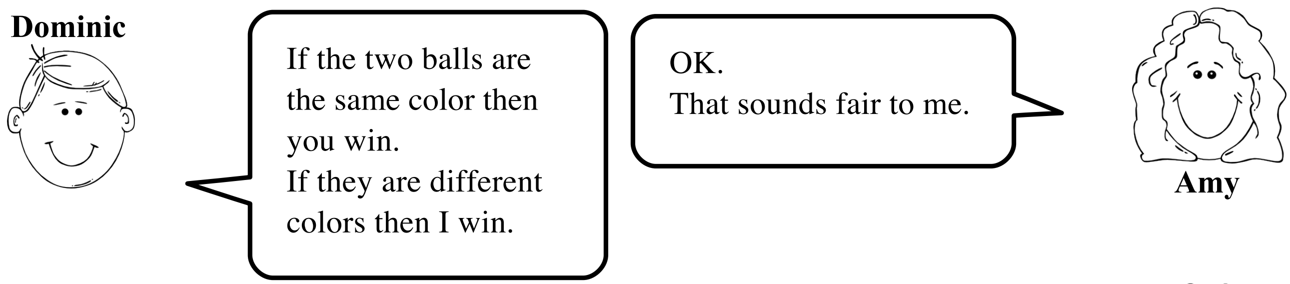
This shows that the number of balls in the bag of each color must be two consecutive triangle numbers from the set: 1, 3, 6, 10, 15, 21, 36 ... .

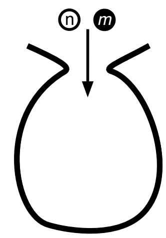
We do not expect many students to reach this level of generalization algebraically, though they may recognize the patterns.

A Fair Game

Dominic has made up a simple game.
Inside a bag he places nine balls. These balls are either black or white.
He then shakes the bag.

He asks Amy to take two balls from the bag without looking.





Amy is right, Dominic has made the game fair.

How many white balls (n) and how many black balls (m) has Dominic put in the bag?

Fully explain your answer.

Hints: You could think about likely combinations of black and white balls and then check to see if they would make the game fair.

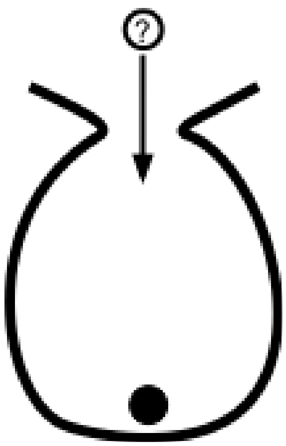
Or you could use algebra to work out the number of black and white balls in the bag.

Make It Fair

A.

1 black ball.

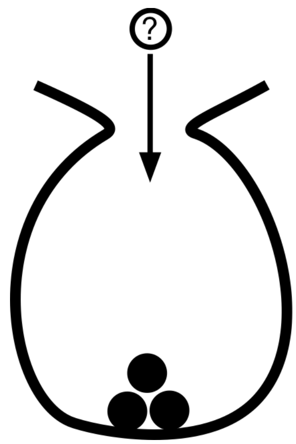
How many white balls for a fair game?



B.

3 black balls.

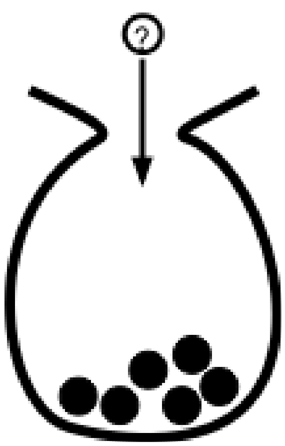
How many white balls for a fair game?



C.

6 black balls.

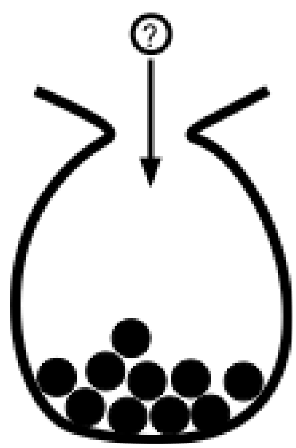
How many white balls for a fair game?



D.

10 black balls.

How many white balls for a fair game?

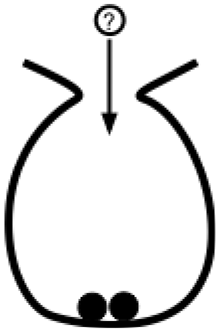


Make It Fair: Extension Task

E.

2 black balls.

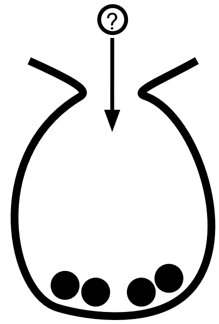
How many white balls for a fair game?



F.

4 black balls.

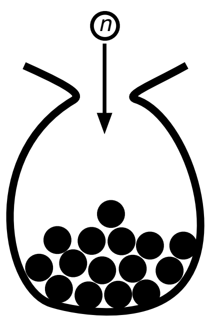
How many white balls for a fair game?



G.

15 black balls.

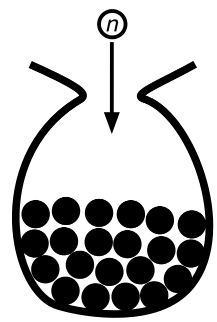
How many white balls for a fair game?



H.

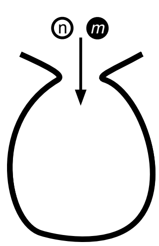
21 black balls.

How many white balls for a fair game?



I.

***m* black balls and *n* white balls.**
If the game is fair, what formulae can you say about *m* and *n*?

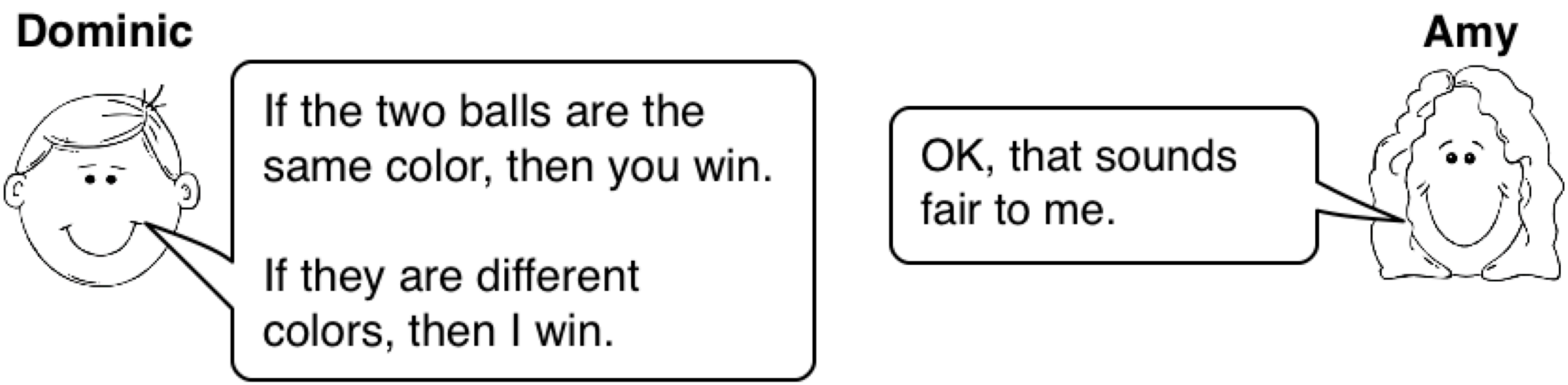


Make it Fair

- Dominic has made up a simple game.

- Inside a bag are a number of black balls and a number of white balls.
- He shakes the bag.

- He asks Amy to take two balls from the bag without looking.



Working Together

- Your task is to figure out how many white balls to add to the bag to make the game fair.
- There may be more than one answer for each card.
- Just choose some number of white balls. If that doesn't make the game fair, think about why it doesn't. Then try to find a different solution.
- It is important that everyone in your group understands and agrees with the solution. If you don't, explain why. You are responsible for each other's learning.
- Prepare a presentation of your problem and your work on the poster.

Mathematics Assessment Project
CLASSROOM CHALLENGES

This lesson was designed and developed by the
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It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

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