PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Inscribing and Circumscribing Right Triangles

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Beta Version

Inscribing and Circumscribing Right Triangles

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to use geometric properties to solve problems. In particular, it will help you identify and help students who have difficulty:

- Decomposing complex shapes into simpler ones in order to solve a problem.
- Bringing together several geometric concepts to solve a problem.
- Finding the relationship between radii of inscribed and circumscribed circles of right triangles.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
7. Look for and make use of structure.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-CO Make geometric constructions.
G-SRT Define trigonometric ratios and solve problems involving right triangles.
G-C Understand and apply theorems about circles.
A-CED Create equations that describe numbers or relationships.

INTRODUCTION

This lesson is designed to enable students to develop strategies for describing relationships between right triangles and the radii of their inscribed and circumscribed circles.

- Before the lesson students attempt the problem individually. You review their work and formulate questions for students to answer in order to improve their solutions.
- At the start of the lesson students work alone answering your questions.
- Students then work in groups in a collaborative discussion of the same task. In the same small groups, students are given some sample solutions to evaluate and comment on.
- In a whole class discussion, students explain and compare the alternative solution strategies they have seen and used.
- Finally, students review what they have learnt.

MATERIALS REQUIRED

- Each student will need a copy of each of: *Inscribing and Circumscribing Right Triangles, Circle Theorems*, the questionnaire *How Did You Work*, a mini-whiteboard, a pen, and an eraser.
- Each group of students will need an enlarged copy of *Inscribing and Circumscribing Right Triangles* and *Sample Responses to Discuss*.
- Compasses, rules, pencils and protractors can be given to students if requested.
- There is also a projector resource to help with whole class discussions.

TIME NEEDED

15 minutes before the lesson and a 100-minute lesson (or two 50-minute lessons). Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Inscribing and Circumscribing Right Triangles* (15 minutes)

Give this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work, and to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.

Give each student a copy of the task *Inscribing and Circumscribing Right Triangles,* and the sheet, *Circle Theorems*.

Have compasses, pencils, rules, and protractors available for students who request them.

Briefly introduce the task and help the class to understand the problem and its context. Make sure everyone understands the problem and the words 'inscribed' and 'circumscribed'.

Read through the task and try to answer it as carefully as you can.

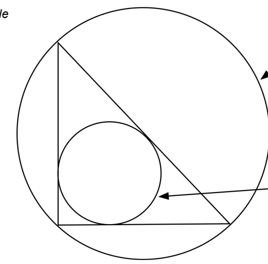
Show all your working so that I can understand your reasoning.

Inscribing and Circumscribing Right Triangles

The circle that passes through all three vertices of a triangle is called circumscribed circle.

The largest circle that fits inside a triangle is called an inscribed circle.

Not to scale



1. Figure out the radii of the circumscribed and inscribed circles for a right triangle with sides 5 units, 12 units and 13 units.
   Show and justify every step of your reasoning.
   The sheet of circle theorems may help you.

   ....

2. Use mathematics to explain carefully how you can figure out the radii of the circumscribed and inscribed circles of a right triangle with sides of *any* length: a, b and c (where c is the hypotenuse)

   ....

It is important that, as far as possible, students are allowed to answer the questions without assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Students who sit together often produce similar answers, and then when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

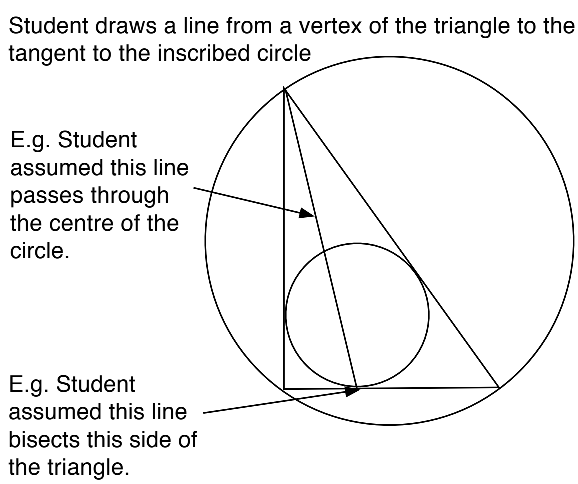
Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores, and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next two pages. These have been drawn from common difficulties observed in trials of this unit.

We recommend that you write a selection of questions on each student's work. If you do not have time, select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson.

Common issues	Suggested questions and prompts
Student has difficulty getting started (Q1)	• What do you know about the angles or lines in the diagram? How can you use what you know? What do you need to find out? • Which circle theorems are most relevant? • Can you add any helpful construction lines to your diagram? What do you know about these lines?
Student explanation is unclear, irrelevant or incomplete (Q1 and Q2) For example: The student has written correct facts, such as the hypotenuse is the diameter of the circumscribed circle, but does not use theorems to support their answers. Or: The student has not explained why angles are 90° or why lengths are equal.	• Would someone unfamiliar with your type of solution easily understand your work? • Can you use one of the circle theorems to describe why the hypotenuse is the diameter of the circumscribed circle? • What circle theorems can you use to support your explanation? • What do the circle theorems tell you about tangents?
Student makes incorrect assumptions based on perception (Q1 and Q2) For example: The student assumes that as the radius of the circumscribed circle is half the length of a tangent then the radius of the inscribed circle is half the length of one of its tangents. The student may write: "radius inscribed circle is $5 \div 2 = 2.5$ units." (Q1)  	• Have you made any assumptions? Can you use the circle theorems to support your work? • Why is the radius of the circumscribed circle half of 13? Does it make sense to apply this explanation to figure out the radius of the inscribed circle? [No]

Common issues	Suggested questions and prompts
Student uses geometric construction to find the relationships (Q2) For example: The student constructs two or three triangles and their inscribed and circumscribed circles and finds the answers by measuring.	• What are the advantages/disadvantages of constructing the shapes? • What do you notice about the hypotenuse of the triangle and the diameter of the circumscribed circle? Will this be the case for all right angled triangles with vertices on the circumference of a circle? Why? • What do you notice about the points where the triangle touches the inscribed circle? Can you use what you've noticed to figure out the radius of an inscribed circle of any right triangle?
Student does not attempt to generalize (Q2) For example: Student has successfully figured out the radii for a specific circle, but makes no attempt to generalize the method.	• What do you notice about the lengths of the sides of the triangle and the radius of the circumscribed circle? Do you know for sure this is always the case? • Can you use your method from Question 1 for a right triangle that has sides of length a, b, and c?
Student correctly answers all the questions	• Can you solve the problem using a different method? • How could you adapt your method if the triangle was not a right triangle, but scalene?

SUGGESTED LESSON OUTLINE

Individual work (5 minutes)

Give students back their work. Give each student a mini-whiteboard, a pen, and an eraser.

If you did not add questions to individual pieces of work, write your list of questions on the board. Students select questions appropriate to their own work, and spend a few minutes answering them. Some students may struggle to identify which questions they should be considering. If this is the case, it may be helpful to give students a printed version of the list of questions so that you can highlight the ones that you want them to focus on.

Begin the lesson by briefly reintroducing the problem. You may want to show the class Slide P-1 of the projector resource.

Recall what we were looking at in a previous lesson. What was the task about?

Today we are going to work together to try to improve your initial attempts at this task.

I have had a look at your work, and I have some questions I would like you to think about.

On your own, carefully read through the questions I have written. I would like you to use the questions to help you to think about ways of improving your own work.

Use your mini-whiteboards to make a note of anything you think will help to improve your work.

Collaborative activity (30 minutes)

Organize the class into small groups of two or three students.

Give each group an enlarged copy of the task *Inscribing and Circumscribing Right Triangles*.

Deciding on a Strategy

Ask students to share their ideas about the task and plan a joint solution.

I want you to share your work with your group.

Take turns to explain how you did the task and how you now think it could be improved.

Listen carefully to any explanation. Ask questions if you don't understand or agree with the method. (You may want to use some of the questions I have written on the board.)

I want you to plan a joint approach that is better than your separate solutions.

Once students have evaluated the relative merits of each approach ask them to write their strategy on the second side of the hand out.

Slide P-2 of the projector resource, *Planning a Joint Solution*, summarizes these instructions.

Implementing the Strategy

Students are now to write their joint solution on the front side of the hand out.

While students work in small groups you have two tasks, to note different student approaches to the task and to support student problem solving.

In particular, note any common mistakes. Are students making any incorrect assumptions? What circle theorems are students using? Are students constructing the diagram? Note if and how students use algebra. You can then use this information to focus a whole class discussion towards the end of the lesson. Attend also to students' mathematical decisions. Do they track their own progress? Do they notice if they have chosen a strategy that does not seem to be productive? If so, what do they do?

Try not to make suggestions that move students towards a particular approach to this task. Instead, ask questions that help students clarify their thinking, promote further progress and encourage students to check their work and detect errors. You may want to use some of the questions in the *Common issues* table to support your own questioning or, if the whole class is struggling on the same issue, write relevant questions on the board and hold a brief whole class discussion. You could also give any struggling students one of the sample responses.

Reviewing the strategy

As students finish working on the problem give them the review questionnaire *How Did You Work?*

Ask students to answer **Questions 1** and **2** of this questionnaire.

Whole-class discussion (10 minutes)

You may want to hold a brief whole-class discussion. Have students solved the problem using a variety of methods? Have you noticed some interesting ways of working or some incorrect methods? If so, you may want to focus the discussion on these. Equally, if you have noticed different groups using similar strategies but making different assumptions you may want to compare solutions.

Collaborative analysis of *Sample Responses to Discuss* (30 minutes)

This task gives students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy. Research has shown that exposing students to multiple perspectives of a solution can deepen their understanding of the mathematics.

After students have had sufficient time to solve the problem, give each group Luke's, Natalie's and Logan's sample responses. If there is not time for all groups to look at all three solutions, be selective about what you give out. For example, groups that have successfully completed the task using one method will benefit from looking at different approaches. Other groups that have struggled with a particular approach may benefit from seeing a student version of the same strategy.

If your students are not familiar with geometric constructions, you may want to withhold Aiden's work. We found in trials of this lesson that on reviewing Aiden's response some students tried to construct the diagram, this left little time to evaluate the three remaining student responses.

Have compasses, pencils, rules, and protractors available for students who request them.

> *Imagine you are the teacher and have to assess this work.*
>
> *Answer the questions below each response.*

Slide P-3 of the projector resource, *Evaluating Sample Responses*, describes how students should work together.

To encourage students to do more than check to see if the answer is correct, ask them to answer the questions below each sample piece of work. Encourage students to focus on the math of the student work, not whether the student has neat writing etc.

During the small group work, support the students as before. Check to see which of the explanations students find more difficult to understand. Note similarities and differences between the sample approaches and those the students took in the small group work.

Extension Task

For Natalie's and Logan's work students are asked to figure out a general formula for the radius of an inscribed circle of any right triangle. These formulas look different. You may want to ask students to investigate whether these formulas represent the same radius.

What can you figure out if you assume the formulas are equal?

The equivalence of Logan's and Natalie's formulae is shown as follows:

If:

$$\frac{ab}{a+b+c} = \frac{a+b-c}{2}$$

Then:

$$2ab = (a+b+c)(a+b-c)$$

$$2ab = a^2 + ab - ac + ab + b^2 - bc + ac + bc - c^2$$

$$2ab = a^2 + b^2 - c^2 + 2ab$$

$$a^2 + b^2 = c^2$$

This shows that if the two equations are equivalent, then the triangle is right-angled, which is indeed the case.

Only Logan's method and formula will work in a situation where there is not a right triangle.

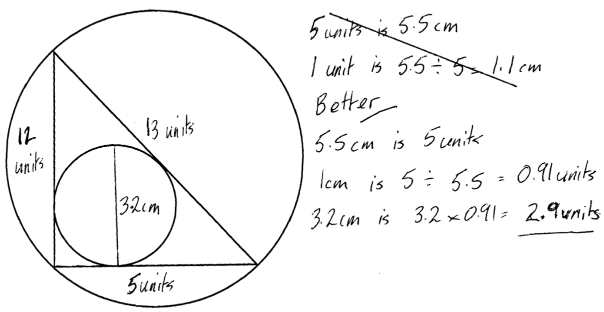
Whole-class discussion: Comparing different approaches (15 minutes)

Now hold a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on parts of the small groups tasks students found difficult. Ask the students to compare the different solution methods.

Which approach did you like best? Why?

Which approach did you find most difficult to understand?

To critique the different strategies use the questions on the sheets *Sample responses to discuss* and Slides P4 - 7 of the projector resource.

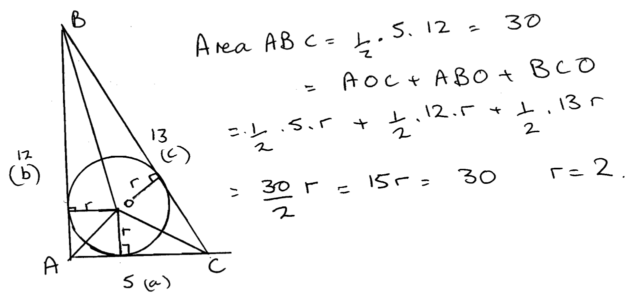


Luke's method Q2

Luke has treated the diagram as a scale drawing. Luke has figured out the diameter not the radius.

The diagram is not to scale. To figure out the radius of the inscribed circle Luke needs to draw a scale diagram.

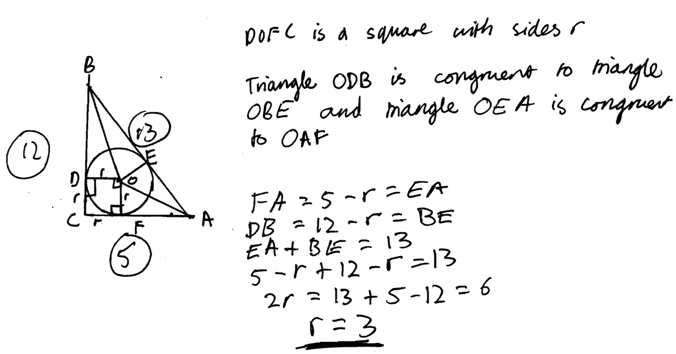
Using a scale drawing is a useful introduction to the problem, but it will not provide a general solution and so cannot be taken further.



Logan's method Q2

Logan has split the right triangle ABC into three smaller ones: AOB, BOC and AOC. Logan correctly explains his method: add together the areas of the 3 smaller triangles. This total is equal to the area of the triangle ABC.

Logan correctly labels right angles in the diagram but does not explain why they are 90º (tangents to the circle.)



Natalie's method Q2

Natalie correctly states that DOFC is a square with sides of length r, but does not support this with an explanation. (Segments CD and CF are tangents and are therefore congruent. ∠ACB = 90°.)

Natalie correctly states the congruency of the triangles, but again there is no supporting explanation.

For example, Δ BOD is congruent to Δ BOE because OB is common, OD = OE = r, DB = BE (If two segments from the same exterior point are tangent to a circle, then the segments are congruent.)

Natalie's method is correct, however she makes a mistake manipulating the equation.

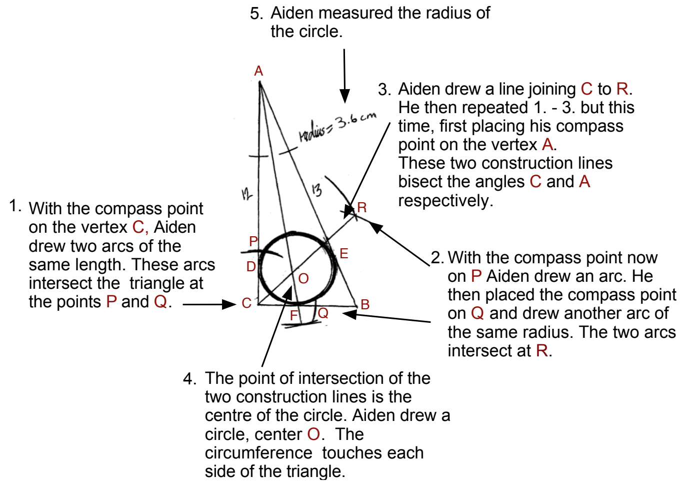
The correct solution is:

$2r = -13 + 5 + 12$

$2r = 4$

$r = 2.$

Aiden's method Q2



Aiden's method Q2

Aiden has correctly identifies the center of the circle.

Aiden uses circle theorems to construct the inscribed circle because:

If two segments from the same exterior point are tangent to a circle, then the segments are congruent.

Therefore:

AD = AE, EB = FB, CF = DC.

Δ DOC is congruent to Δ EOF (DO = OE; AD = AE, and CO is a common.)

Therefore the line that bisects the angle at this point is equal distance from the two tangents and passes through the center of the inscribed circle (tangents to circles.)

Aiden has measured the diameter not the radius.

Constructing the circles and the triangles can be a useful introduction to the problem, but it will not provide a general solution and so cannot be taken further.

If you have time, you may want to ask students to investigate applying the general formulae derived from Logan's and Natalie's work to any triangle. [Only Logan's method can be used.]

Review solutions to Inscribing and Circumscribing Right Triangles (10 minutes)

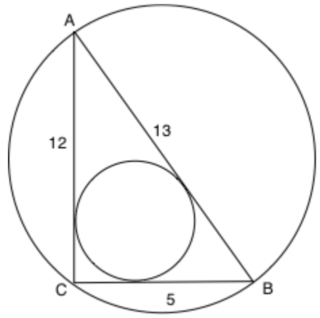
Ask students to complete the review questionnaire. Some teachers set this task as homework.

The questionnaire should help students monitor and review their progress during and at the end of an activity.

If you have time you may also want to ask your students to read through their original solution and using what they have learned, attempt the task again.

SOLUTIONS

1.



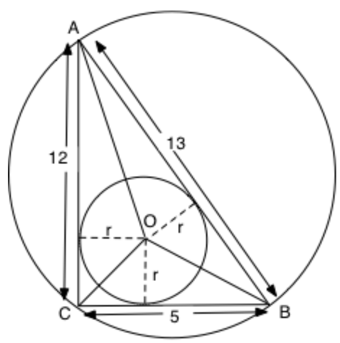
Radius of circumscribed circle

$\angle BCA = 90°$.

If an angle is inscribed in a circle, then its measure is half the measure of its intercepted arc (or central angle.)

Therefore the central angle is 180º. It then follows that AB is a diameter.

Radius of circumscribed circle = $13 \div 2 = 6.5$



Radius of inscribed circle - method 1

The lines AC, BC and CA are all tangents to the circle and so are perpendicular to the radii at the point they touch the circle.

Area Δ ABC = area Δ BCO + area Δ ACO+ area Δ ABO

$$\frac{1}{2} \times 12 \times 5 = \frac{1}{2} \times 5 \times r + \frac{1}{2} \times 12 \times r + \frac{1}{2} \times 13 \times r$$

$$30 = \frac{1}{2}(5r + 12r + 13r)$$

$$30 = 15r$$

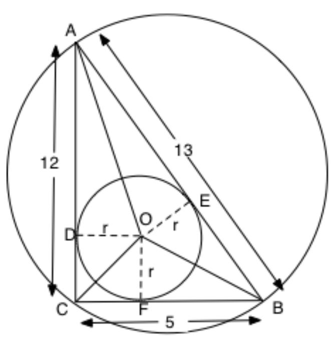
$$r = 2$$

Radius of inscribed circle - method 2

If two segments from the same exterior point are tangent to a circle, then the segments are congruent. Therefore:

AD = AE, EB = FB, CF = DC.

Δ DOC is congruent to Δ COF
(DO = OF = r, DC = FC, and CO is a common.)



Therefore $\angle OCF = 90° \div 2 = 45°$.

$\tan \angle OCF = \tan 45° = 1 = r \div CF$

$CF = r = DC$

It follows that:

$5 = CF + FB$

$5 = r + FB$

$12 = AD + DC = AD + r$

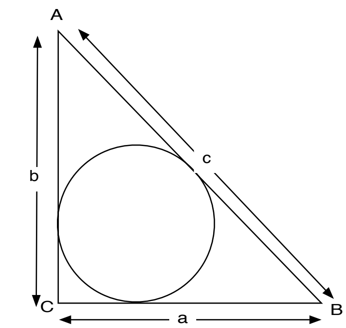
$13 = AE + EB$

$5 + 12 - 13 = r + FB + AD + r - AE - EB$

$4 = 2r$ (FB = EB; AD = AE)

$r = 2$

2. Describe carefully a method for working out the radius of the circumscribed and inscribed circle of a right triangle with sides of length a, b and c.



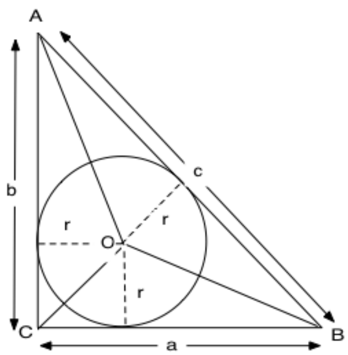
Radius of circumscribed circle

$\angle BCA = 90°$.

If an angle is inscribed in a circle, then its measure is half the measure of its intercepted arc (or central angle.)

Therefore the central angle is 180º. It then follows that AB is a diameter.

Radius of circumscribed circle = $\frac{c}{2}$



Radius of an inscribed circle

Logan's method:

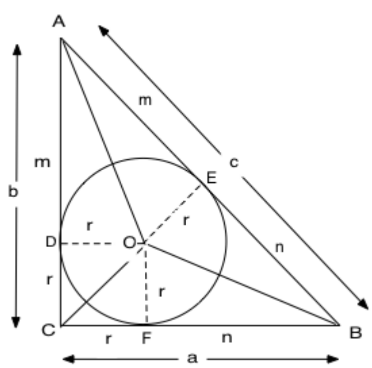
The lines AC, BC and CA are all tangents to the circle and so are perpendicular to the radii at the point they touch the circle.

Area Δ ABC = area Δ BCO + area Δ ACO+ area Δ AB

$$\frac{1}{2}ab = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

$$\frac{1}{2}ab = \frac{1}{2}r(a + b + c)$$

$$r = \frac{ab}{a + b + c}$$



Natalie's method:

If two segments from the same exterior point are tangent to a circle, then the segments are congruent. Therefore:

AD = AE; EB = FB; CF = DC

Δ DOC is congruent to Δ COF (DO = OF, DC = FC, and CO is a common.)

$\angle BCA = 90º$
Therefore $\angle OCF = 90º \div 2 = 45º$.

Tan $\angle OCF$ = tan 45º = 1 = $r \div$ CF

CF = r = DC

It follows that:

$a = n + r$ $b = r + m$ $c = m + n$

$a + b - c = (n + r) + (r + m) - (m + n) = 2r$

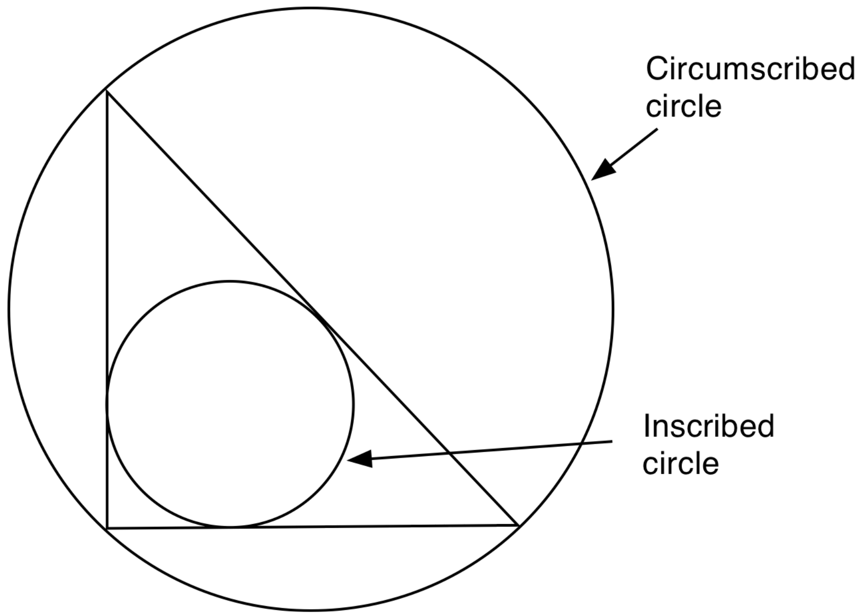
$$r = \frac{a + b - c}{2}$$

Inscribing and Circumscribing Right Triangles

The circle that passes through all three vertices of a triangle is called circumscribed circle.

The largest circle that fits inside a triangle is called an inscribed circle.

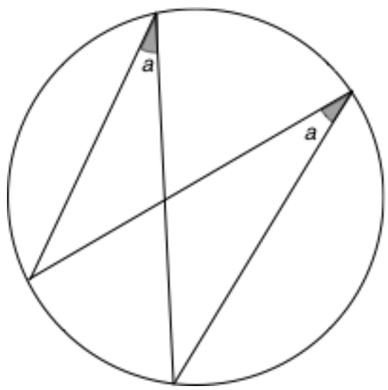
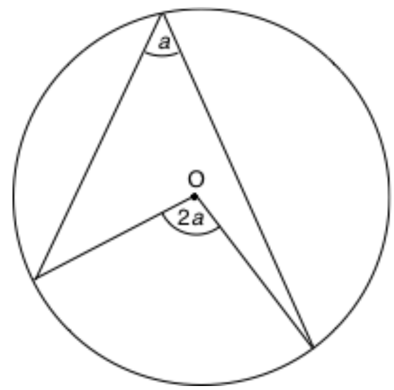
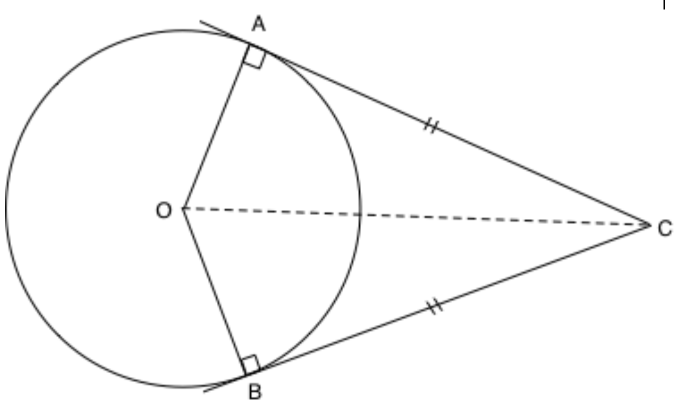
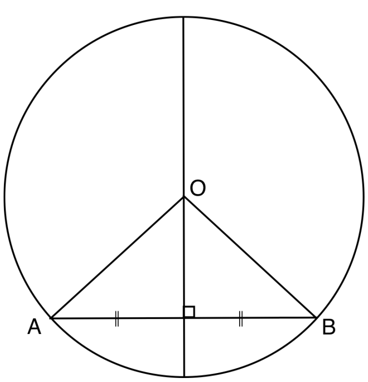
This diagram is not drawn to scale



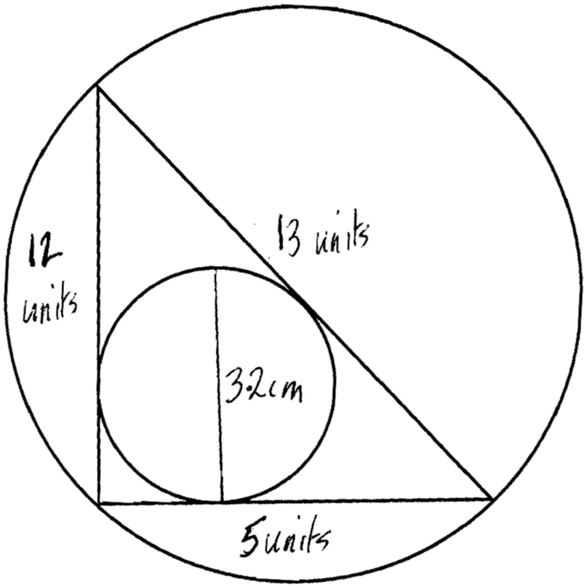
1. Figure out the radii of the circumscribed and inscribed circles for a right triangle with sides 5 units, 12 units and 13 units.
   Show and justify every step of your reasoning.
   The sheet of circle theorems may help you.

2. Use mathematics to explain carefully how you can figure out the radii of the circumscribed and inscribed circles of a right triangle with sides of *any* length: a, b and c (where c is the hypotenuse)

Circle Theorems

Theorem 1 If inscribed angles of a circle intercept the same arc, then they are congruent.  	**Theorem 2** If an angle is inscribed in a circle, then its measure is half the measure of its intercepted arc (or central angle.) 
Theorem 3 If two segments from the same exterior point are tangent to a circle, then the segments are congruent.  	**Theorem 4** If a diameter of a circle is perpendicular to a chord, then the diameter bisects the chord. 

Sample Responses to Discuss - Luke



~~5 units is 5.5cm~~

~~1 unit is 5.5 ÷ 5 = 1.1 cm~~

Better

5.5cm is 5units

1cm is 5 ÷ 5.5 = 0.91units

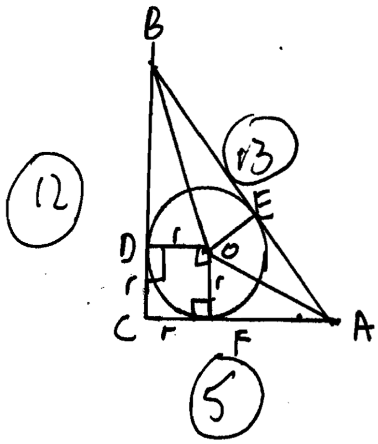
3.2cm is 3.2 × 0.91 = 2.9units

Why did Luke change his method?

How can Luke improve his work?

Can the improved method be used to answer **all** the task questions?

Sample Responses to Discuss - Natalie



DOFC is a square with sides r

Triangle ODB is congruent to triangle OBE and triangle OEA is congruent to OAF

$FA = 5 - r = EA$
$DB = 12 - r = BE$
$EA + BE = 13$
$5 - r + 12 - r = 13$
$2r = 13 + 5 - 12 = 6$
$\underline{r = 3}$

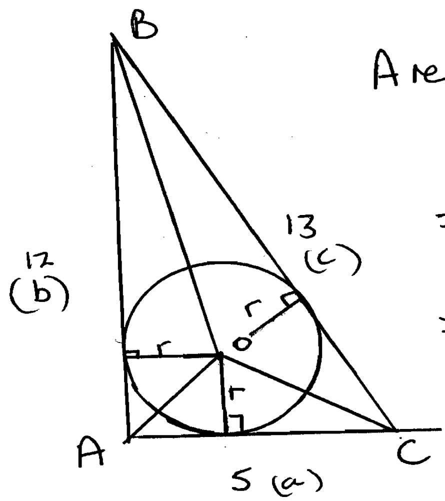
Check Natalie's work carefully and correct any errors you find.

What isn't clear about Natalie's work?

Which circle theorems has Natalie used?

Use the same method to figure out the radius of the inscribed circle when the sides of the triangle are a, b, and c.

Sample Responses to Discuss - Logan



$$\text{Area } ABC = \tfrac{1}{2} \cdot 5 \cdot 12 = 30$$

$$= AOC + ABO + BCO$$

$$= \tfrac{1}{2} \cdot 5 \cdot r + \tfrac{1}{2} \cdot 12 \cdot r + \tfrac{1}{2} \cdot 13 r$$

$$= \frac{30}{2} r = 15r = 30 \qquad r = 2.$$

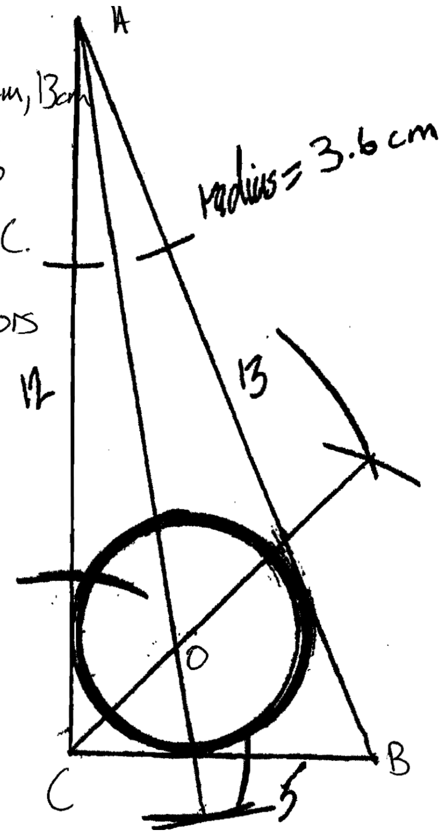
Check Logan's work carefully and correct any errors you find.

What isn't clear about the work?

Use the same method to figure out the radius of the inscribed circle when the sides of the triangle are a, b, and c.

Sample Responses to Discuss - Aiden

I constructed a
right triangle 5cm, 12cm, 13cm
I used a compass to
bisect angles A and C.
The 2 angle bisectors
intersected at O.
I drew the circle
center O.



Check Aiden’s work carefully and correct any errors you find.

Explain how Aiden uses circle theorems to construct the inscribed circle.

What are the problems with using construction when answering **all** the questions in the task?

How Did You Work?

Mark the boxes and complete the sentences that apply to your work.

1. Our solution is similar to one of the sample responses ☐

   Our group solution was better because:

   ---

2. In our solution we showed where we used the following circle theorems:

   ---

3. Our solution is similar to one of the sample responses ☐

   Our solution is similar to (add name of the student)

   I prefer our solution / the student's solution (circle)

   because

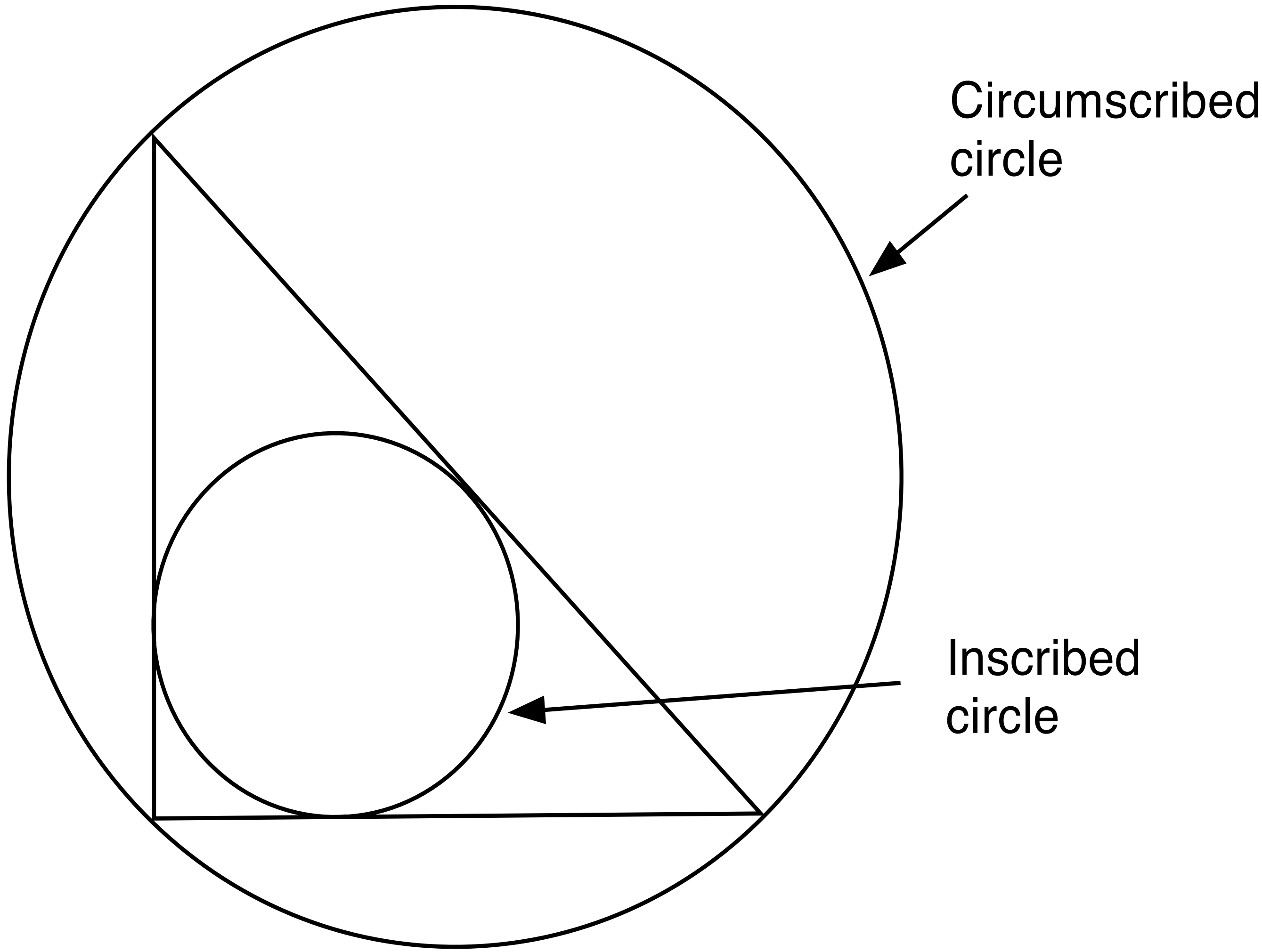
   OR

   Our solution is different from all of the sample responses ☐

   because

4. What advice would you give a student new to this task about potential pitfalls?

Inscribing and Circumscribing Right Triangles



Planning a Joint Solution

1. Take turns to explain how you did the task and how you now think it could be improved.

2. Listen carefully to explanations.
   - Ask questions if you don't understand.

   - Discuss with your partners:
     - What you like/dislike about your partners' math.
     - Any assumptions your partner has made.
     - How their work could be improved.

3. Once everyone in the group has explained their solution, plan a joint approach that is better than each of the separate solutions.
   - On the second side of your sheet of paper write a couple of sentences, outlining your plan.

Evaluating Sample Responses

1. Imagine you are the teacher and have to assess the student work.

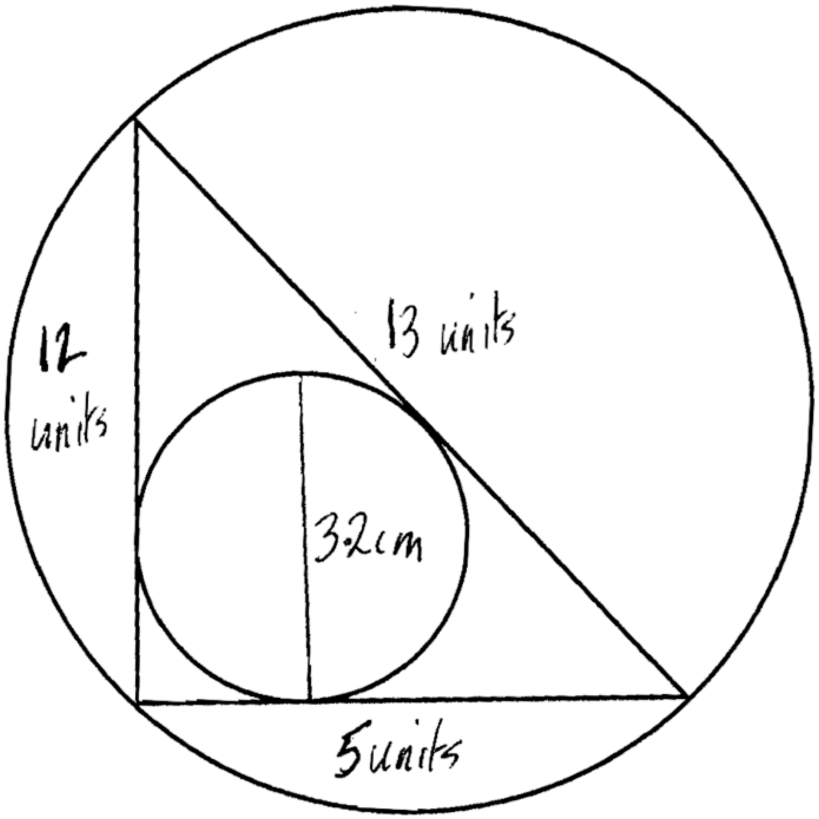
2. Take it in turns to work through a students' solution.
   - Write your answers on your mini-whiteboards.

3. Explain your answer to the rest of the group.

4. Listen carefully to explanations.
   - Ask questions if you don't understand.

5. Once everyone is satisfied with the explanations, write the answers below the students' solution.
   - Make sure the student who writes the answers is not the student who explained them.

Sample Response to Discuss - Luke



~~5 units is 5.5 cm~~

~~1 unit is $5.5 \div 5 = 1.1$ cm~~

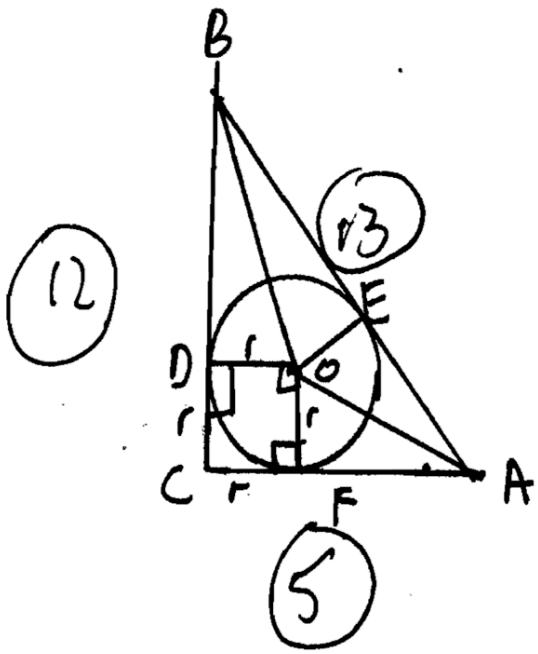
Better

5.5 cm is 5 units

1cm is $5 \div 5.5 = 0.91$ units

3.2cm is $3.2 \times 0.91 =$ 2.9 units

Sample Response to Discuss - Natalie

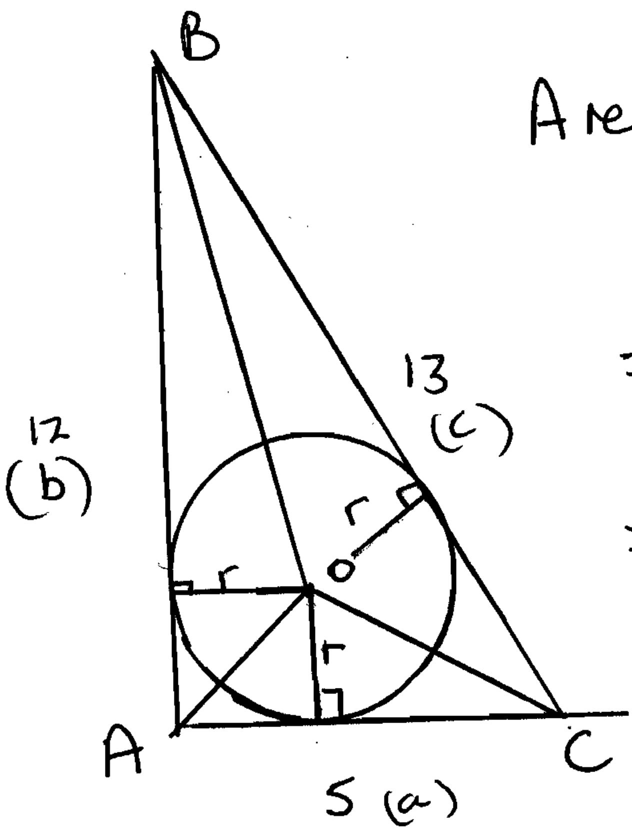


DOFC is a square with sides r

Triangle ODB is congruent to triangle OBE and triangle OEA is congruent to OAF

$FA = 5 - r = EA$
$DB = 12 - r = BE$
$EA + BE = 13$
$5 - r + 12 - r = 13$
$2r = 13 + 5 - 12 = 6$
$\underline{r = 3}$

Sample Response to Discuss - Logan



$$\text{Area } ABC = \frac{1}{2} \cdot 5 \cdot 12 = 30$$

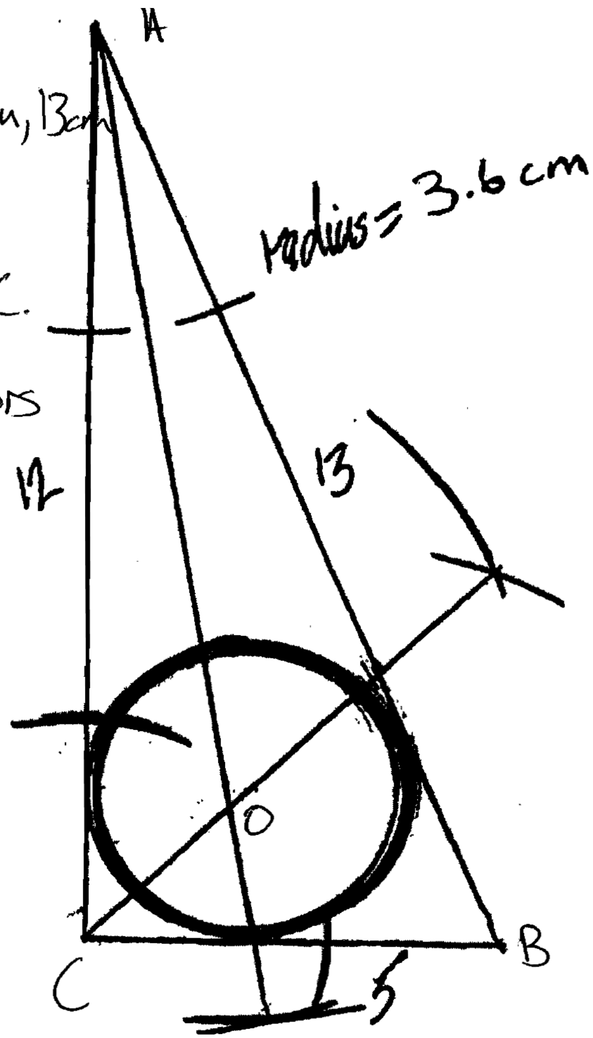
$$= AOC + ABO + BCO$$

$$= \frac{1}{2} \cdot 5 \cdot r + \frac{1}{2} \cdot 12 \cdot r + \frac{1}{2} \cdot 13 r$$

$$= \frac{30}{2} r = 15r = 30 \qquad r = 2.$$

Sample Response to Discuss - Aiden

I constructed a
right triangle 5cm, 12cm, 13cm
I used a compass to
bisect angles A and C.
The 2 angle bisectors
intersected at O.
I drew the circle
center O.



Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
at the
University of Nottingham
Malcolm Swan, Nichola Clarke, Clare Dawson, Sheila Evans
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Carol Hill, and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of
Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee