CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Solving Linear Equations in Two Variables

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

For more details, visit: http://map.mathshell.org
© 2014 MARS, Shell Center, University of Nottingham
Please do not distribute outside schools participating in the trials

Solving Linear Equations in Two Variables

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to formulate and solve problems using algebra and, in particular, to identify and help students who have the following difficulties:

- Solving a problem using two linear equations with two variables.
- Interpreting the meaning of algebraic expressions.

COMMON CORE STATE STANDARDS

This lesson involves *mathematical content* in the standards from across the grades, with emphasis on:

A-CED: Create equations that describe numbers or relationships.
A-REI: Solve systems of equations.

This lesson involves a range of *mathematical practices*, with emphasis on:

2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

INTRODUCTION

This lesson is structured in the following way:

- Before the lesson, students work individually on the assessment task *Notebooks and Pens*. You then review their work and create questions for students to answer in order to improve their solutions.
- During the lesson, students work individually on a task that requires them to interpret and solve two equations in two variables. Students then compare and discuss their solutions in small groups.
- In the same small groups, students evaluate some sample solutions of the same task.
- In a whole-class discussion, students explain and compare the alternative solution strategies they have seen and used.
- Finally, in a follow-up lesson, students use what they have learned to revise their work on *Notebooks and Pens*.

MATERIALS REQUIRED

- Each individual student will need two copies of the assessment task *Notebooks and Pens*, and a copy of the lesson task *School Fair*.
- Each small group of students will need a large blank sheet of paper for making a poster, and an enlarged copy of the *Sample Student Work*.
- Graph paper should be kept in reserve and used only when requested.
- Projector resources are provided as support.

TIME NEEDED

Approximately fifteen minutes before the lesson, a one-hour lesson, and ten minutes in a follow-up lesson (or for homework). Timings given are only approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Notebooks and Pens* (15 minutes)

Have the students do this task in class or for homework a day or more before the lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.

Give each student a copy of *Notebooks and Pens*.

Introduce the task briefly and help the class to understand the problem.

> *Read through the questions, and answer them carefully.*
>
> *Show all your work, so that I can understand your reasoning.*

It is important that students are allowed to answer the questions without assistance, as far as possible.

Notebooks and Pens

A store sells pens at $2 and notebooks at $5.

n = number of notebooks sold.

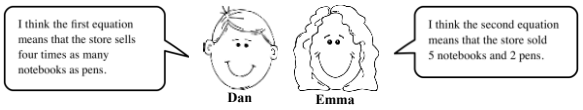
p = number of pens sold.

The following equations are true:

$4n = p$

$5n + 2p = 39$

Here is what Dan and Emma think the equations mean:



Is Dan correct?

If you think Dan is wrong, explain the mistake and explain what *you* think $4n = p$ means.

Is Emma correct?

If you think Emma is wrong, explain the mistake and explain what *you* think $5n + 2p = 39$ means.

Figure out for yourself the number of pens and the number of notebooks sold in the store.

Number of pens sold = ... Number of notebooks sold = ...

Students should not worry too much if they cannot understand or do everything, because there will be a lesson using a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table below. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- Write one or two questions on each student's work, or
- Give each student a printed version of your list of questions, and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these on the board when you return the work to the students in the follow-up lesson.

Common issues:	Suggested questions and prompts:
Student assumes that the letter stands for an object not a number For example: The student says that the statements are correct. Or: The student realizes the equations are incorrect, but is unable to explain why.	• What does the letter p represent? • Write the equation as a sentence. Does your sentence match what Dan/Emma said? • If $n = 3$, what would p equal in the first equation? Which is greater: n or p? • Are there more notebooks than pens? How can you tell from the equation?
Student only uses one equation For example: The student finds a value or values for n and p that fits one equation but not the other, such as $n = 1$ and $p = 4$ for the first equation.	• For this equation, is there another possible pair of values for n and p? And another? How do you know which value is correct? • How can you check that your values for n and p work for both equations?
Student produces unsystematic guess and check work For example: The student works out three or four seemingly unconnected combinations of values for n and p.	• What is a sensible value to try for n (or p)? Why? • Can you organize your work in a table?
Student provides poor explanation For example: The student presents the work as a series of unexplained numbers and/or calculations.	• Would someone unfamiliar with your type of solution easily understand your work? • Have you explained how you arrived at your answer?
Student makes algebraic mistakes For example: The student makes a mistake when manipulating the algebra in the equations.	• How can you check that your answer is correct?

SUGGESTED LESSON OUTLINE

Individual work: *School Fair* (10 minutes)

Give each student the task sheet *School Fair*. Help students to understand the problem, and explain the context of the task briefly.

Spend ten minutes on your own answering these questions.

What does "simultaneously" mean?

Show all your work on the sheet.

Students who sit together often produce similar answers and, when they come to compare their work, they have little to discuss.

For this reason we suggest that, when students do this task individually, you ask them to move to different seats. Then, for the collaborative task, allow them to return to their usual places. Experience has shown that this produces more profitable discussions.

School Fair

Joe is mixing nuts and raisins to sell at a school fair.
He buys nuts in 4 pound bags and raisins in 1 pound bags.

x = number of bags of nuts he buys
y = number of bags of raisins he buys.

The following equations are true:

$3x = y$
$4x + y = 70$

1. Explain in words the meaning of each equation.

2. Find two pairs of values for x and y that satisfy the first equation.

3. Find two pairs of values for x and y that satisfy the second equation.

4. Find pairs of values for x and y that satisfy both equations *simultaneously*.

Collaborative small-group work: *School Fair* (15 minutes)

Organize the class into small groups of two or three students.

Show and explain to students Slide P-1 of the projector resource:

Sharing Individual Solutions

1. Take turns to share with your partner(s) your own individual solution to the task.
2. Listen carefully to each other, asking questions if you don't understand.
3. Notice any similarities or differences between the methods described.

Explain to students that this activity will enable them to decide which approach to collaboratively pursue.

Once students have had chance to discuss their work, hand out to each group a sheet of poster paper.

In your groups agree on the best method for completing the task.

Produce a poster that shows a joint solution that is better than your individual work.

Students should now have another go at the task, but this time they will combine their ideas.

Throughout this activity, encourage students to articulate their reasoning, justify their choices mathematically, and question the choices put forward by others.

As students work you have two tasks, to note student approaches to their work, and to support their thinking.

Make a note of student approaches to the task

Listen and watch students carefully. In particular, notice how students make a start on the task, where they get stuck and how they overcome any difficulties.

How do students choose to tackle this task? Notice the variety in approaches. Notice any common errors. You can use this information to focus your questioning in the whole-class discussion towards the end of the lesson.

Support students working as a group

As students work on the task support them in working together. Try not to make suggestions that prompt students towards a particular answer. Instead, ask questions to help students clarify their thinking.

You may find that some students interpret the letters as "nuts" and "raisins" rather than the number of nuts and number of raisins. For example, they may say things like:

> "$3x = y$ means three times as many raisins as nuts."
> "$4x + y = 70$ means 4(lbs of) nuts plus 1(lb of) raisins equals 70(lbs)."
> "There are 70 bags."

The following questions and prompts may be helpful for both students struggling with the task and those making quick progress:

What do the letters x and y represent?

Replace x and y in this equation by words and now say what the equation means.

Are there more bags of raisins or more bags of nuts? How do you know?

Do you have any values for x and y that work for the first equation? How can you check to see if they also work for the second one? If these don't fit, what other values for x and y can you use?

Why have you chosen these values for x and y?

Suppose there are 5 bags of nuts, so x = 5.
From the first equation, how many bags of raisins are there are there? [15.]
From the second equation, how many bags of raisins are there? [50.]
There cannot be both 15 and 50 bags of raisins!

Can you find a value for x that will give the same answer in both cases?

How can you check that your answer is right?

Can you use these equations to calculate the amount of bags Joe has?

If the whole class is struggling on the same issue, you may want to write a couple of questions on the board and organize a brief whole-class discussion. You could also ask students who performed well in the assessment to help struggling students.

Collaborative analysis of *Sample Student Work* (20 minutes)

When all groups have made a reasonable attempt, ask them to put their work to one side. Give each group enlarged copies of the *Sample Student Work.* This task will give students the opportunity to discuss and evaluate possible approaches to the task, without providing a complete solution strategy.

Ideally, all groups will review all four pieces of work. However, if you are running out of time, choose just two solutions for **all** groups to analyze, using what you have learned during the lesson about what students find most difficult.

Display and explain to students Slide P-2 of the projector resource:

Sample Responses to Discuss

1. Read each piece of sample student work carefully.
2. Try to understand what they have done. You may want to add annotations to the work to make it easier to follow.
3. Think about how the work could be improved. Take turns explaining your thinking to your partner.
4. Listen carefully and ask clarifying questions.
5. When your group has reached its conclusions, write your answers to the questions.

During this small-group work, support the students as before. Also, check to see which of the explanations students find more difficult to understand. Note similarities and differences between the sample approaches and those the students used in the group work.

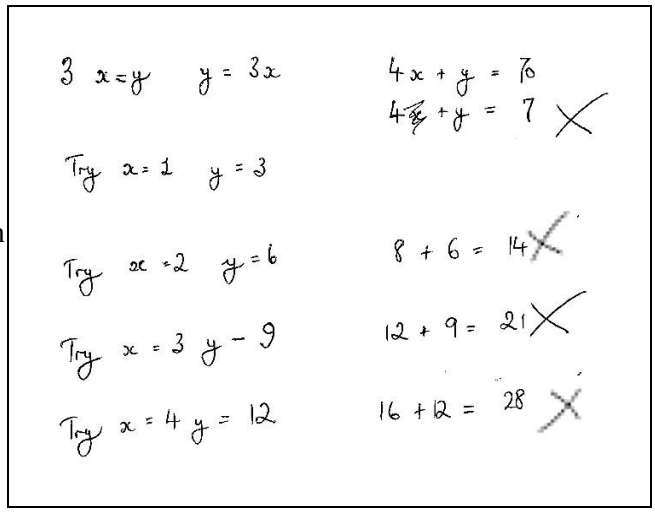
Ava used "guess and check" with both equations

Strengths: Her work is systematic and easy to follow.

Weaknesses: Her method is inefficient and, although it is systematic, she has not reflected on each answer to determine the next set of values to check.

Her lack of progress leads to her abandoning the task.

Ava could add an explanation about her solution method.

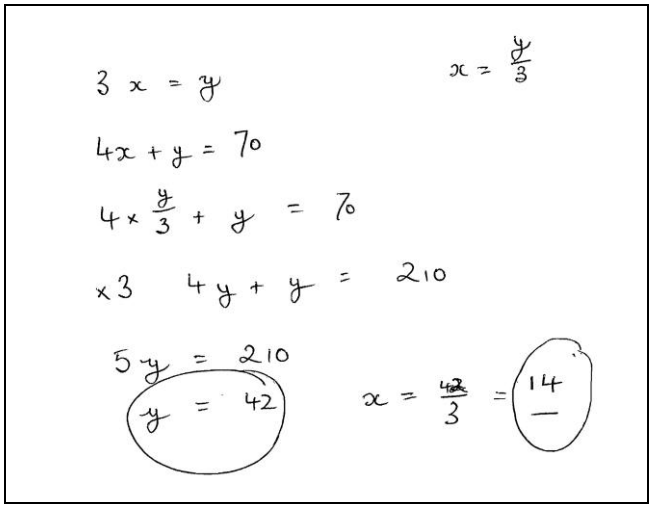


Joe used a substitution method

Strengths: This is an efficient method.

Weaknesses: Joe failed to multiply all the terms on the left-hand side of the equation by three, so he obtained an incorrect answer.

If Joe had substituted $3x$ for y into the second equation the solution would have been very straightforward.



Ethan used an elimination method

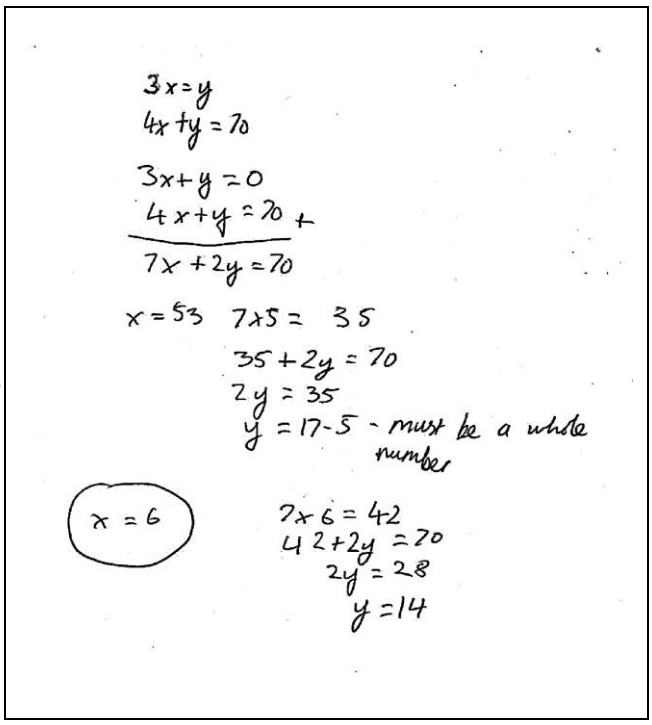
Strengths: This method can work if equations are manipulated carefully.

Weaknesses: Ethan makes a mistake when rearranging the first equation. Consequently, when the two equations are added together, a variable is not eliminated, but instead Ethan has created an equation with two variables.

Ethan briefly used guess and check. This gives many solutions. Ethan has simply opted to figure out two solutions. Both answers are incorrect. Ethan has not explained his working or why he was happy with the second set of values.

If the first equation had been $3x + y = 0$, what would still be wrong with Ethan's method?

Would this method ever obtain just one solution?



Mia used a graphical approach

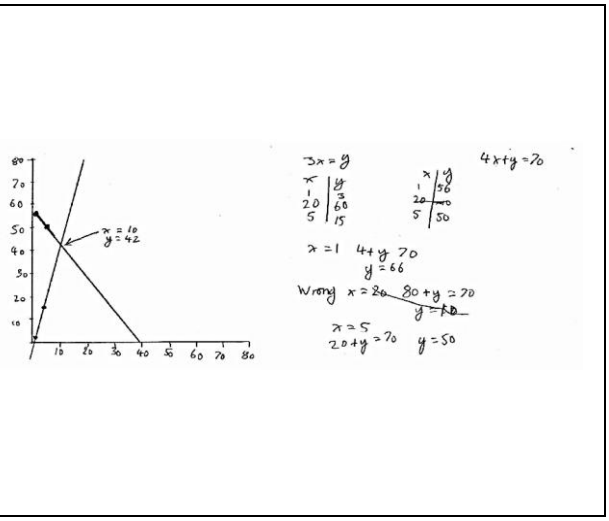
Strengths: This method can work.

Weaknesses: In this case a graphical approach is not a very efficient strategy.

Mia has made an error in her second table: $y = 66$ not 56.

Mia could have used the co-ordinates (20,−10) to help plot the second line. There are no labels on either axis. The scale of Mia's graph means that the lines are not plotted accurately.

Was Mia right to abandon (20, −10) as a point to be used to plot the second line?



Whole-class discussion: comparing different approaches (15 minutes)

Hold a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on those parts of the task that the students found difficult.

It may be helpful to display Slides P-3 to P-6 during this discussion.

Compare 'error-free', improved versions of the methods otherwise students may focus on just the errors or lack of explanations rather than whether the method was 'fit for purpose'.

Which piece of work did you find easiest/most difficult to understand? Why was that?
Are there any unsuitable methods? Please explain.
Which method is the least efficient? Please explain.
Which method did you like best? Why?
How does your method compare to the sample student work? Are there similarities/differences?

Follow-up lesson: Reviewing the assessment task (15 minutes)

Have students do this task at the beginning of the next lesson if you do not have time during the lesson itself. Some teachers like to set this task for homework.

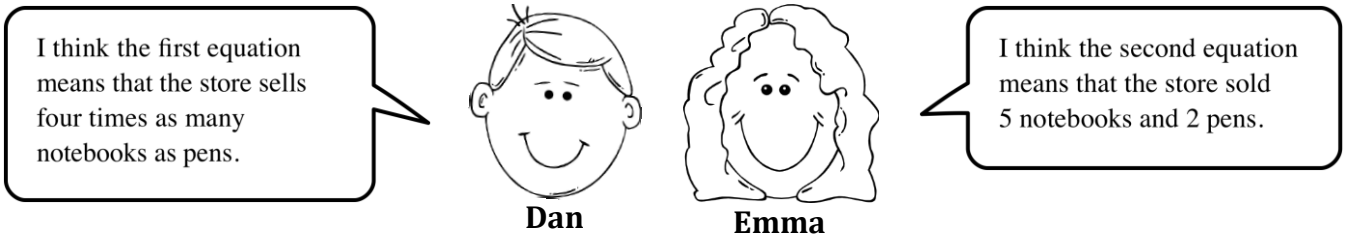
Return the students' individual work on the assessment task *Notebooks and Pens* along with a second blank copy of the task sheet.

> *Look at your original responses and think about what you have learned this lesson.*
> *Using what you have learned, try to improve your work.*

If you have not written questions on individual pieces of work then write your list of questions on the board. Students are to select from this list only the questions appropriate to their own work.

SOLUTIONS

Assessment task: Notebooks and Pens



Dan is incorrect:

Dan has misinterpreted n to mean, "notebooks sold" rather than "*the number of* notebooks sold."

So he has read the equation "$4n = p$" as "there are four notebooks sold for every single pen sold."

The equation actually means, "4 times the number of notebooks sold equals the number of pens sold," or "the store sells four times more pens than notebooks."

Emma is incorrect:

Emma has also misinterpreted n to mean "notebooks" rather than "*the number of* notebooks."

In the second statement, $5n$ does not mean, "there are 5 notebooks." It means "5 times the number of notebooks."

Since each notebook costs \$5, $5n$ gives you the amount of money taken from selling notebooks, and since each pen costs \$2, $2p$ gives you the amount of money taken from selling pens. So $5n + 2p = 39$ means that \$39 was taken altogether from selling notebooks and pens at these prices. However, the equation does not, in isolation, tell you how many notebooks or pens were sold.

Using the first equation to substitute $4n$ for p in the second equation gives $n = 3$ and $p = 12$.

3 notebooks and 12 pens were sold.

School Fair

1. The number of bags of raisins is three times the number of bags of nuts.
   Four times the number of bags of nuts plus the number of raisins totals 70 (the total weight of all nuts and raisins is 70 pounds).
2. Possible values: (4, 12) or (8, 24).
3. Possible values: (12, 22) or (7, 42).
4. $x = 10, y = 30$

Notebooks and Pens

A store sells pens at \$2 and notebooks at \$5.

n = number of notebooks sold.

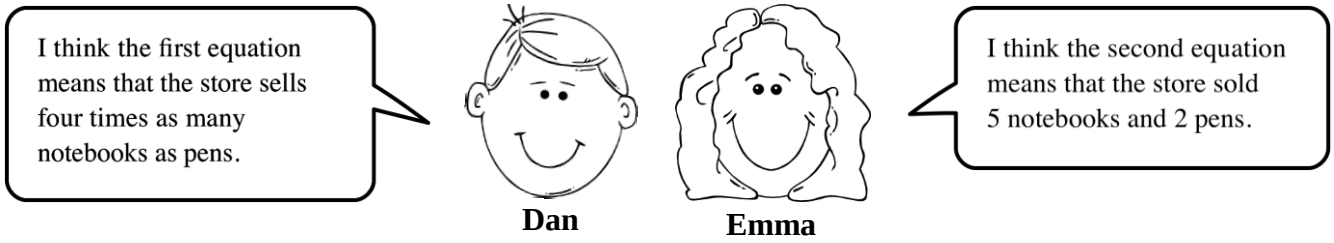
p = number of pens sold.

The following equations are true:

$4n = p$

$5n + 2p = 39$

Here is what Dan and Emma think the equations mean:



Are Dan and Emma correct?

If you think Dan is wrong, explain the mistake and explain what *you* think the equation means.

If you think Emma is wrong, explain the mistake and explain what *you* think the equation means.

© 2014 MARS, Shell Center, University of Nottingham

Figure out for yourself the number of pens and the number of notebooks sold in the store.

© 2014 MARS, Shell Center, University of Nottingham

School Fair

Joe is mixing nuts and raisins to sell at a school fair.
He buys nuts in 4-pound bags and raisins in 1-pound bags.

x = number of bags of nuts he buys
y = number of bags of raisins he buys.

The following equations are true:

$3x = y$
$4x + y = 70$

1. Explain in words the meaning of each equation.

2. Find two pairs of values for x and y that satisfy the first equation.

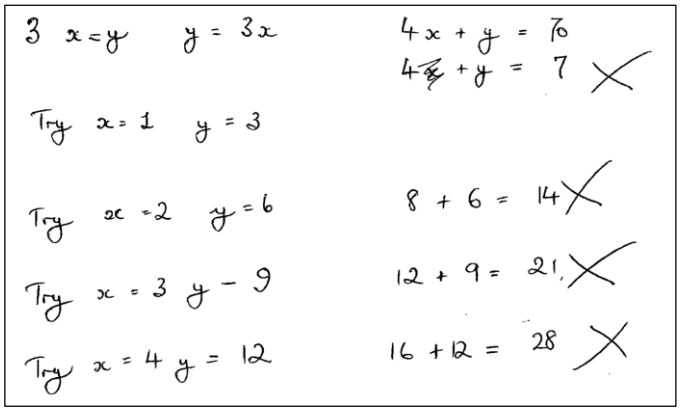
3. Find two pairs of values for x and y that satisfy the second equation.

4. Find pairs of values for x and y that satisfy both equations *simultaneously*.

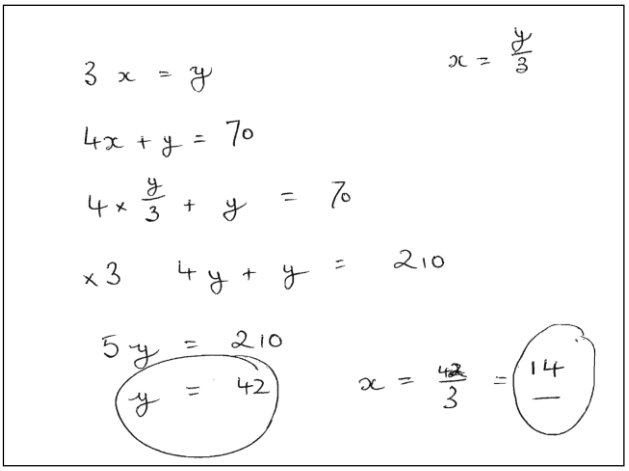
© 2014 MARS, Shell Center, University of Nottingham

Sample Student Work

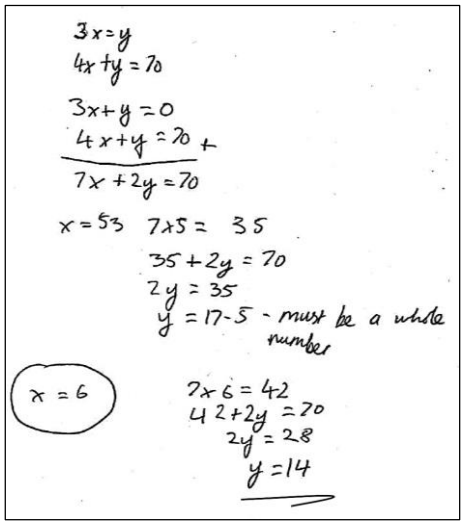
Ava



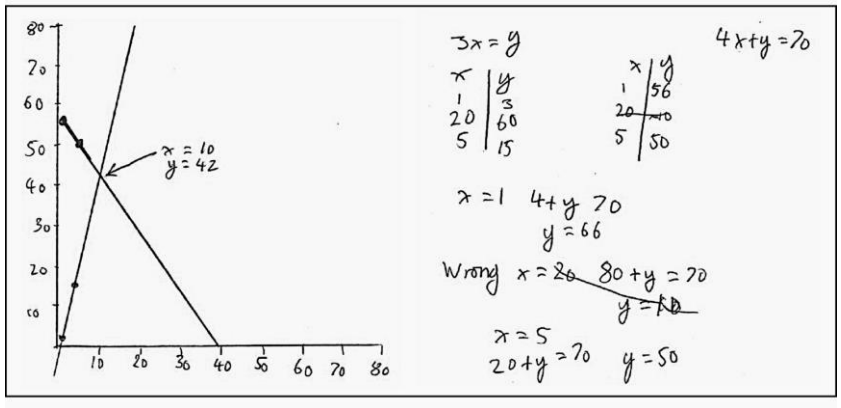
Joe



Ethan



Mia



For each piece of work:

- What method was used? Is it clear? Is it accurate?
- Are any methods unsuitable for the task? Explain your answer.
  If you think a method is suitable, make changes to the work in order to improve it.

© 2014 MARS, Shell Center, University of Nottingham

Sharing Individual Solutions

1. Take turns to share with your partner(s) your own individual solution to the task.

2. Listen carefully to each other, asking questions if you don't understand.

3. Notice any similarities or differences between the methods described.

Sample Responses to Discuss

1. Read each piece of sample student work carefully.

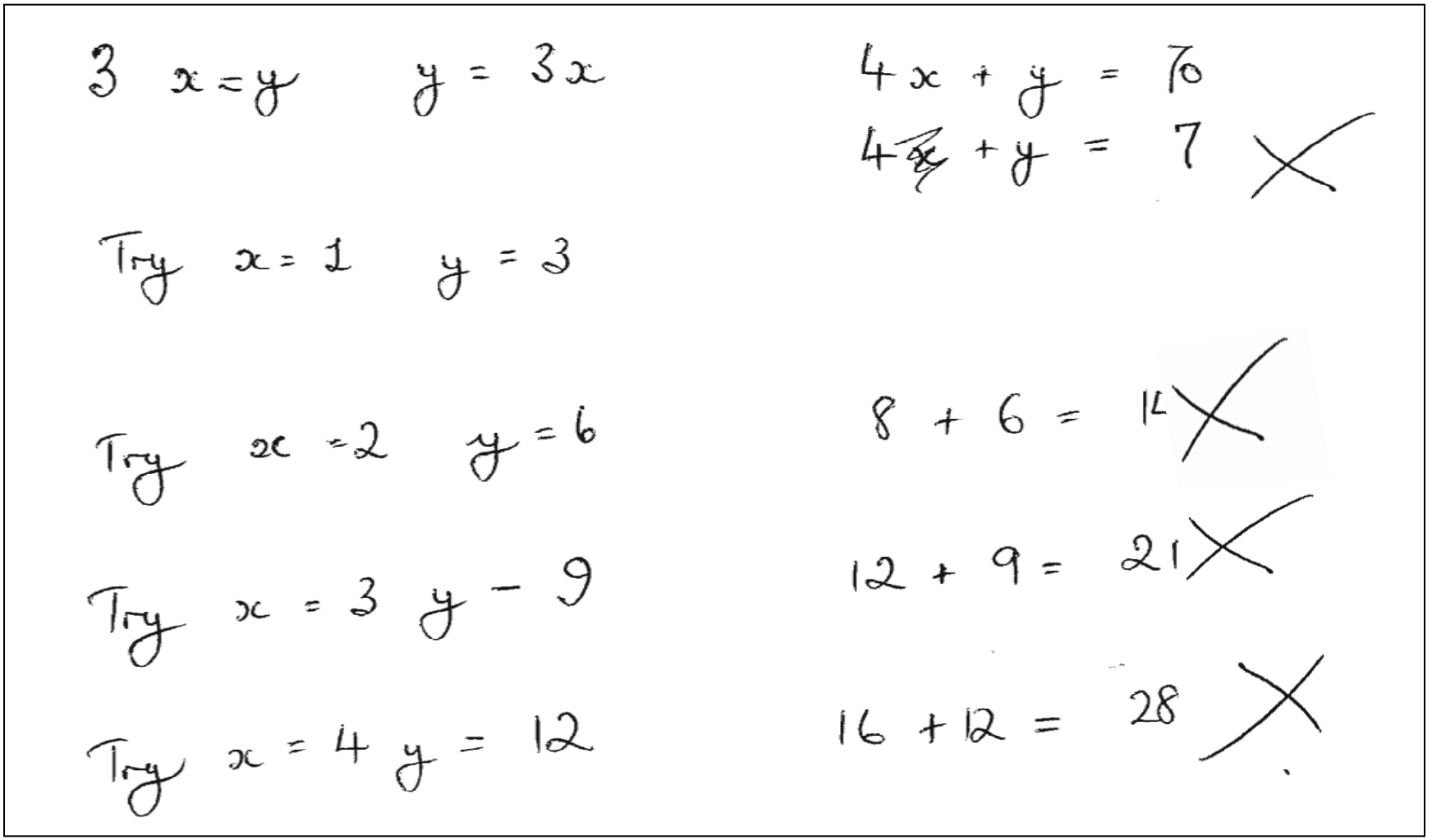
2. Try to understand what they have done. You may want to add annotations to the work to make it easier to follow.

3. Think about how the work could be improved. Take turns explaining your thinking to your partner.

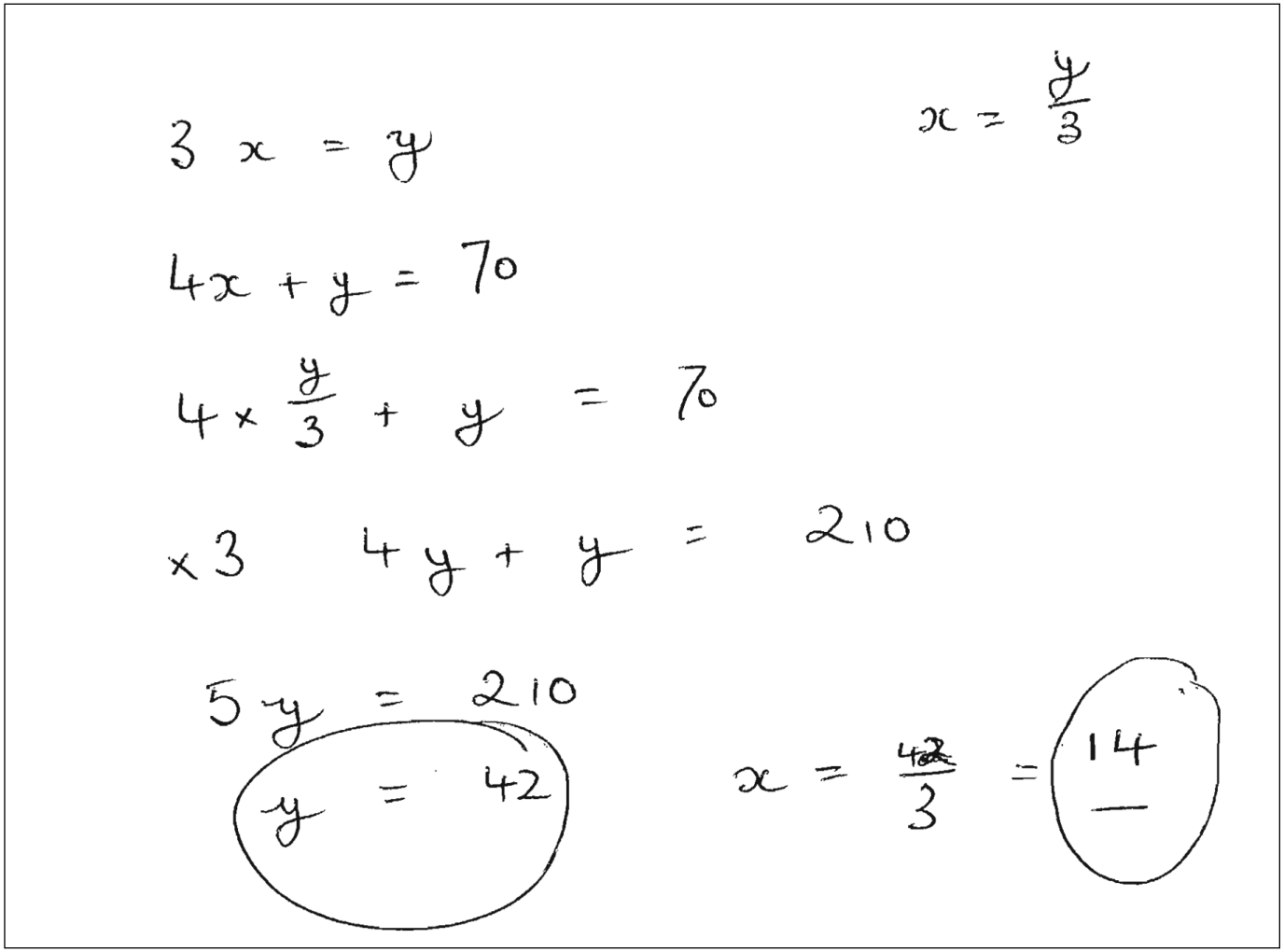
4. Listen carefully and ask clarifying questions.

5. When your group has reached its conclusions, write your answers to the questions.

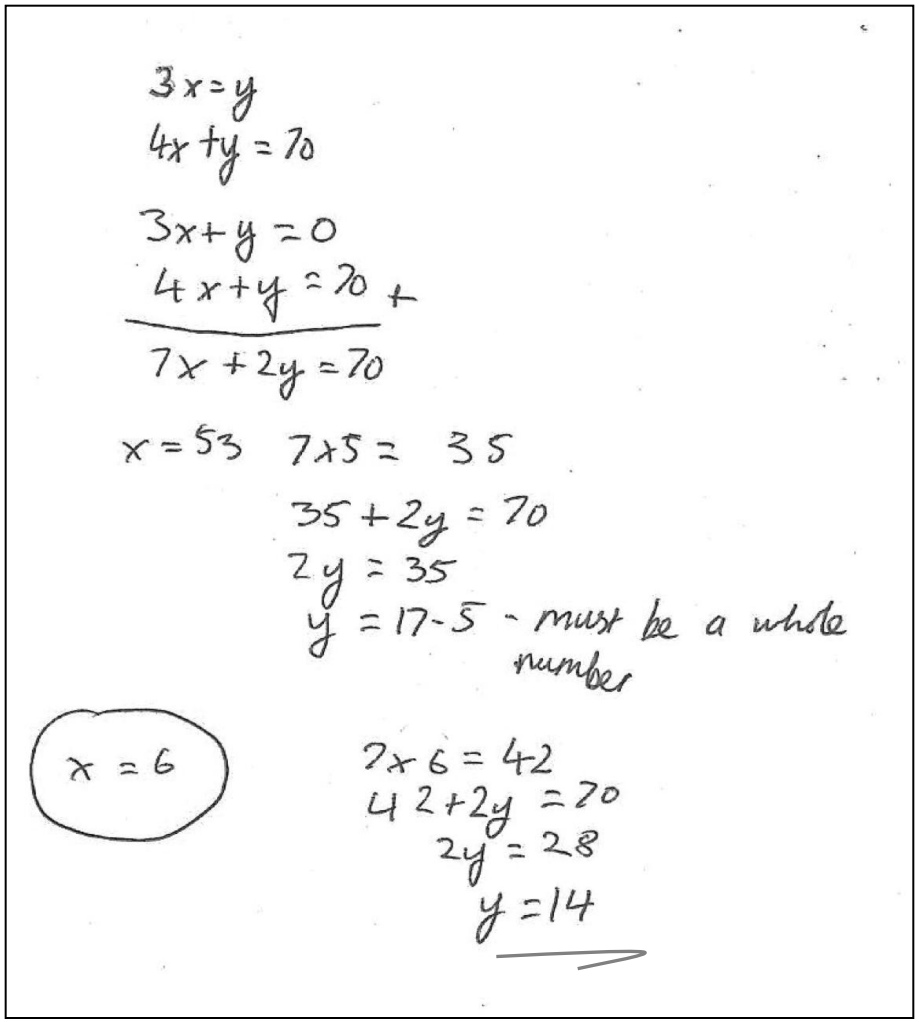
Sample Student Work: Ava



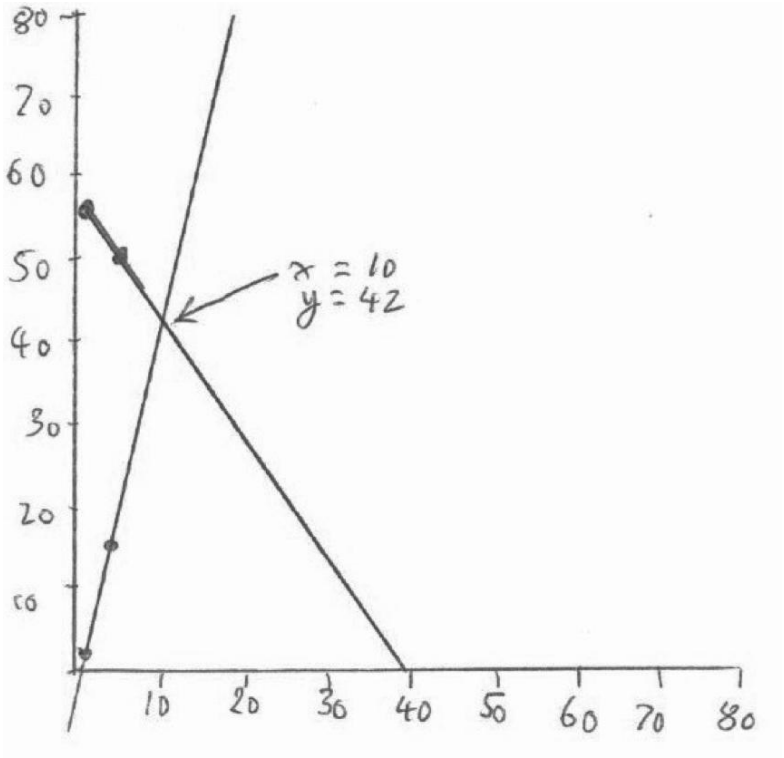
Sample Student Work: Joe



Sample Student Work: Ethan



Sample Student Work: Mia



$3x = y$

x	y
1	3
20	60
5	15

$4x + y = 70$

x	y
1	56
20	~~10~~
5	50

$x = 1 \quad 4 + y \; 70$

$y = 66$

Wrong $x = 20 \quad 80 + y = 70$

$y = 10$

$x = 5$

$20 + y = 70 \quad y = 50$

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
at the
University of Nottingham
Malcolm Swan, Clare Dawson, Sheila Evans,
Marie Joubert and Colin Foster
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Mathew Crosier, Nick Orchard and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of
Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee

© 2014 MARS, Shell Center, University of Nottingham
This material may be reproduced and distributed, without modification, for non-commercial purposes,
under the Creative Commons License detailed at http://creativecommons.org/licenses/by-nc-nd/3.0/
All other rights reserved.
Please contact map.info@mathshell.org if this license does not meet your needs.