



## Lesson 18: Looking More Carefully at Parallel Lines

### Student Outcomes

- Students learn to construct a line parallel to a given line through a point not on that line using a rotation by  $180^\circ$ . They learn how to prove the alternate interior angles theorem using the parallel postulate and the construction.

### Lesson Notes

Lesson 18 is important! It may take you two days to cover all of the information in this lesson. Hiding underneath the ideas in this lesson are most of the reasons why reflection and translation can be defined based upon the geometry assumptions (see Lesson 33). (For example, we use only the geometry assumptions to construct a parallel line through a given point in Example 4.) While these ideas are “hiding under the surface,” don’t worry about them. Instead, concentrate students’ attention on the big geometry assumption of this lesson: the parallel postulate.

We have already defined parallel lines, and in the missing angles problems we have used (and perhaps discussed casually) the following two ideas:

- Suppose a transversal intersects a pair of lines. If a pair of alternate interior angles is equal in measure, then the pair of lines are parallel.
- (A form of the Parallel Postulate) Suppose a transversal intersects a pair of lines. If the pair of lines are parallel, then the pair of alternate interior angles are equal in measure.

However, students have probably not made careful distinctions between these thoughts. More likely, they simply remember:

*Suppose a transversal intersects a pair of lines. The pair of lines are parallel if and only if a pair of alternate interior angles are equal in measure.*

Perhaps, they simply associate the ideas in this statement. When they see parallel lines cut by a transversal, they recognize the angle relations. They use the angle relations, when they are pointed out, to recognize parallel lines.

This lesson is designed to help students carefully distinguish between these two ideas. In particular, we will show why there is a need for the parallel postulate as one of our geometric assumptions: *through a given external point there is at most one line parallel to a given line.*

### Classwork

#### Opening Exercise

##### Opening Exercise

Exchange homework problems with a classmate. Solve the problems posed by your classmate while he/she solves yours. Compare your solutions, discuss and resolve any discrepancies. Why were you not asked to rotate a pre-image to obtain the image, but asked only to locate the point of rotation? How did you use perpendicular bisectors in constructing your solutions?

Discussion

Discussion

We say that two lines are *parallel* if they lie in the same plane and do not intersect. Two segments or rays are parallel if the lines containing them are parallel.

Example 1

Example 1

Why is the phrase *in the plane* critical to the definition of parallel lines? Explain and illustrate your reasoning.

*Two lines in (3-dimensional space) are called skew lines if they do not lie in the same plane. In that case, they do not intersect (because if they did, they would share a plane together) and are not parallel (if they were parallel, then they would both have to lie in the same plane).*



In Lesson 7, we recalled some basic facts learned in earlier grades about pairs of lines and angles created by a transversal to those lines. One of those basic facts was:

*Suppose a transversal intersects a pair of lines. The lines are parallel if and only if a pair of alternate interior angles are equal in measure.*

It is our goal in this lesson to prove this theorem using basic rigid motions, geometry assumptions, and a geometry assumption we will introduce in this lesson called the *parallel postulate*. Of all of the geometry assumptions we have given so far, the parallel postulate gets a special name because of the special role it played in the history of mathematics. (Euclid included a version of the parallel postulate in his books, and for 2000 years people tried to show that it wasn't a necessary assumption. Not only did it turn out that the assumption was necessary for Euclidean geometry, but study of the parallel postulate lead to the creation of non-Euclidean geometries.)

The basic fact above really has two parts, which we prove separately:

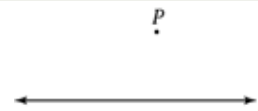
1. Suppose a transversal intersects a pair of lines. If two alternate interior angles are equal in measure, then the pair of lines are parallel.
2. Suppose a transversal intersects a pair of lines. If the lines are parallel, then the pair of alternate interior angles are equal in measure.

The second part turns out to be an equivalent form of the parallel postulate. To build up to the theorem, first we need to do a construction:

Example 2

Example 2

Given a line  $l$  and a point  $P$  not on the line, follow the steps below to rotate  $l$  by  $180^\circ$  to a line  $l'$  that passes through  $P$ :



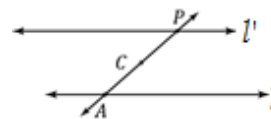
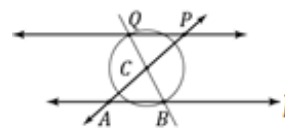
1. Label any point  $A$  on  $l$ .
2. Find the midpoint of segment  $AP$  using a ruler (measure the length of segment  $AP$ , and locate the point that is distance  $AP/2$  from  $A$  between  $A$  and  $P$ ). Label the midpoint  $C$ .

MP.7

MP.7

3. Perform a  $180^\circ$  rotation around center  $C$ . To quickly find the image of  $l$  under this rotation by hand:

- Pick another point  $B$  on  $l$ .
- Draw  $\overline{CB}$ .
- Draw circle: center  $C$ , radius  $CB$ .
- Label the other point where the circle intersects  $\overline{CB}$  by  $Q$ .
- Draw  $\overline{PQ}$ .



4. Label the image of the rotation by  $180^\circ$  of  $l$  by  $l' = R_{C,180}(l)$ .

How does your construction relate the geometry assumption stated above to rotations? Complete the statement below to clarify your observations:

$R_{C,180}$  is a  $180^\circ$  rotation around  $C$ . Rotations preserve lines, therefore  $R_{C,180}$  maps the line  $l$  to the line  $l'$ . What is  $R_{C,180}(A)$ ? P

Example 3

Example 3

The lines  $l$  and  $l'$  in the picture above certainly look parallel, but we do not have to rely on “looks.”

Claim: In the construction above,  $l$  is parallel to  $l'$ .

Proof: We will show that assuming they are not parallel leads to a contradiction. If they are not parallel, then they must intersect somewhere, call that point  $X$ . Since  $X$  is on  $l'$ , it must be the image of some point  $S$  on  $l$  under the  $R_{C,180}$  rotation, i.e.,  $R_{C,180}(S) = X$ . Since  $R_{C,180}$  is a  $180^\circ$  rotation,  $S$  and  $X$  must be the endpoints of a diameter of a circle that has center  $C$ . In particular,  $\overline{SX}$  must contain  $C$ . Since  $S$  is a point on  $l$ , and  $X$  is a different point on  $l'$  (it was the intersection of both lines), we have that  $l = \overline{SX}$  because there is only one line through two points. But  $\overline{SX}$  also contains  $C$ , which means that  $l$  contains  $C$ . However,  $C$  was constructed so that it was not on  $l$ ! This is absurd.

There are only two possibilities for any two distinct lines  $l$  and  $l'$  in a plane: either the lines are parallel or they are not parallel. Since assuming the lines were not parallel lead to a false conclusion, the only possibility left is that  $l$  and  $l'$  were parallel to begin with.

MP.7

Example 4

Example 4

The construction and claim together implies:

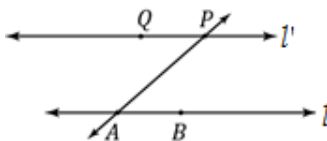
Theorem: Given a line  $l$  and a point  $P$  not on the line, then there exists line  $l'$  that contains  $P$  and is parallel to  $l$ .

This is a theorem we have justified before using compass and straightedge constructions, but now we see it follows directly from basic rigid motions and our geometry assumptions.

Example 5

Example 5

We are now ready to prove the first part of the basic fact above. We have two lines,  $l$  and  $l'$ , and all we know is that a transversal  $\overline{AP}$  intersects  $l$  and  $l'$  such that a pair of alternate interior angles are equal in measure (in the picture to the right, we are assuming  $m\angle QPA = m\angle BAP$ ).



Let  $C$  be the midpoint of  $\overline{AP}$ . What happens if you rotate  $180^\circ$  around the center  $C$ ? Is there enough information to show that  $R_{C,180}(l) = l'$ ?

1. What is the image of the segment  $AP$ ? Segment  $AP$

2. In particular, what is the image of the point  $A$ ?  $P$

3. Why are the points  $Q$  and  $R_{C,180}(B)$  on the same side of  $\overline{AP}$ ?

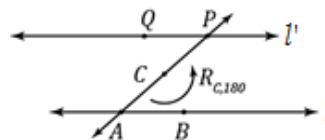
*Sketch of the Answer: The rotation by  $180^\circ$  maps  $\overline{AP}$  to itself because  $\overline{AP}$  contains the center  $C$ . In particular, it can be shown that the rotation maps one half-plane of  $\overline{AP}$  to the other half-plane and vice-versa. Since  $Q$  and  $B$  are in opposite half-planes (by definition of alternate interior angles), and  $B$  and  $R_{C,180}(B)$  are in opposite half-planes,  $Q$  and  $R_{C,180}(B)$  must be in the same half-plane.*

4. What is the image of  $R_{C,180}(\angle BAP)$ ?  $\angle QPA$  Why?

*Because under the rotation, the vertex of  $\angle BAP$  maps to the vertex of  $\angle QPA$ , ray  $AP$  maps to ray  $PA$ , the point  $B$  goes to the same side as  $Q$ . Since  $\angle BAP = \angle QPA$  (by assumption), ray  $AB$  must map to ray  $PQ$ . Thus,  $\angle BAP$  maps to  $\angle QPA$ .*

5. Why is  $R_{C,180}(l) = l'$ ?

*Since ray  $\overline{AB}$  maps to ray  $\overline{PQ}$ ,  $R_{C,180}(l) = R_{C,180}(\overline{AB}) = \overline{PQ} = l'$ .*



We have just proved that a rotation by  $180^\circ$  that takes  $l$  to  $l'$ . Hence, by the claim in Example 3, lines  $l$  and  $l'$  must be parallel, which can be summarized as:

**Theorem.** Suppose a transversal intersects a pair of lines. If a pair of alternate interior angles are equal in measure, then the pair of lines are parallel.

Discussion

Discussion

In Example 5, suppose we had used a different rotation to construct a line parallel to  $l$  that contains  $P$ . Such constructions are certainly plentiful! For example, for every other point  $D$  on  $l$ , we can find the midpoint of segment  $PD$ , and use the construction in Example 2 to construct a different  $180^\circ$  rotation around a different center such that the image of the line  $l$  is a parallel line through the point  $P$ . Are any of these parallel lines through  $P$  different? In other words,

*Can we draw a line other than the line  $l'$  through  $P$  that never meets  $l$ ?*

MP.7

The answer may surprise you—it stumped mathematicians and physicists for centuries. In nature, the answer is that it is sometimes possible, and sometimes not. This is because there are places in the universe (near massive stars, for example) where the model geometry of space is not “plane-like” or flat, but is actually quite curved. To rule out these other types of “strange but beautiful” geometries, we must assume that the answer to the question above is only one line. That choice becomes one of our geometry assumptions:

*(Parallel Postulate) Through a given external point there is at most one line parallel to a given line.*

In other words, we assume that for any point  $P$  in the plane not lying on a line  $\ell$ , every line in the plane that contains  $P$  intersects  $\ell$  except at most one line – the one we call *parallel* to  $\ell$ .

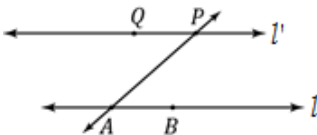
Example 6

Example 6

We can use the parallel postulate to prove the second part of the basic fact:

**Theorem.** Suppose a transversal intersects a pair of lines. If the pair of lines are parallel, then the pair of alternate interior angles are equal in measure.

**Proof:** Suppose that a transversal  $\overline{AP}$  intersects line  $l$  at  $A$  and  $l'$  at  $P$ , and pick and label another point  $B$  on  $l$  and choose a point  $Q$  on  $l'$  on the opposite side of  $\overline{AP}$  as  $B$ . The picture might look like the figure below:



Let  $C$  be the midpoint of segment  $\overline{AP}$ , and apply a rotation by  $180^\circ$  around the center  $C$ . As in previous discussions, the image of  $l$  is the line  $R_{C,180}(l)$  which is parallel to  $l$  and contains point  $P$ . Since  $l'$  and  $R_{C,180}(l)$  are both parallel to  $l$  and contain  $P$ , by the parallel postulate, they must be the same line:  $R_{C,180}(l) = l'$ . In particular,  $R_{C,180}(\angle BAP) = \angle QPA$ . Since rotations preserve angle measures,  $m\angle BAP = m\angle QPA$ , which was what we needed to show.

Discussion

It is important to point out that, although we only proved the alternate interior angles theorem, the same sort of proofs can be done in the exact same way to prove the corresponding angles theorem and the interior angles theorem. Thus, all of the proofs we have done so far (in class and in homework) that use these facts are really based, in part, on our assumptions about rigid motions!

Example 7

Example 7

We end this lesson with a theorem that we will just state, but can be easily proved using the parallel postulate:

**Theorem:** If three distinct lines  $l_1, l_2$ , and  $l_3$  in the plane have the property that  $l_1 \parallel l_2$  and  $l_2 \parallel l_3$ , then  $l_1 \parallel l_3$ . (In proofs, this can be written as, “If two lines are parallel to the same line, then they are parallel to each other.”)

**Final word to teachers:** Students should at least remember that in Euclidean Geometry, two lines are parallel if and only

if alternate interior angles of any transversal are equal in measure, and be able to elaborate on what that means. This one statement includes both the Parallel Postulate and its converse. We can construct parallel lines without the Parallel Postulate, but in a geometry that does not satisfy the Parallel Postulate there are many parallels to a given line through a point not on it. Without the Parallel Postulate, parallel lines are plentiful and we cannot tell much about a line if all we know is that it passes through a point and is parallel to another line.

**Relevant Vocabulary**

**Parallel:** Two lines are *parallel* if they lie in the same plane and do not intersect. Two segments or rays are parallel if the lines containing them are parallel lines.

**Transversal:** Given a pair of lines  $l$  and  $m$  in a plane, a third line  $t$  is a *transversal* if it intersects  $l$  at a single point and intersects  $m$  at a single but different point.

The definition of transversal rules out the possibility that any two of the lines  $l$ ,  $m$ , and  $t$  are the same line.

**Alternate Interior Angles:** Let line  $t$  be a transversal to lines  $l$  and  $m$  such that  $t$  intersects  $l$  at point  $P$  and intersects  $m$  at point  $Q$ . Let  $R$  be a point on  $l$ , and  $S$  be a point on  $m$  such that the points  $R$  and  $S$  lie in opposite half-planes of  $t$ . Then the angle  $\angle RPQ$  and the angle  $\angle PQS$  are called *alternate interior angles* of the transversal  $t$  with respect to  $m$  and  $l$ .

**Corresponding Angles:** Let line  $t$  be a transversal to lines  $l$  and  $m$ . If  $\angle x$  and  $\angle y$  are alternate interior angles, and  $\angle y$  and  $\angle z$  are vertical angles, then  $\angle x$  and  $\angle z$  are *corresponding angles*.

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 18: Looking More Carefully at Parallel Lines

### Exit Ticket

1. Construct a line through the point  $P$  below that is parallel to the line  $l$  by rotating  $l$  by a  $180^\circ$  (using the steps outlined in Example 2).

2. Why is the parallel line you constructed the only line that contains  $P$  and is parallel to  $l$ ?

Exit Ticket Sample Solutions

The following solutions indicate an understanding of the objectives of this lesson:

<p>1. Construct a line through the point <math>P</math> below that is parallel to the line <math>l</math> by rotating <math>l</math> by <math>180^\circ</math> (using the steps outlined in Example 2). <i>Construction should look like the steps in Example 2.</i></p> <p>2. Why is the parallel line you constructed the only line that contains <math>P</math> and is parallel to <math>l</math>? <i>The answer should reference the Parallel Postulate in a meaningful way.</i></p>	
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Problem Set Sample Solutions

The following solutions indicate an understanding of the objectives of this lesson:

<p>1. In the figure, <math>\overline{AL} \parallel \overline{BM}</math>, <math>\overline{AL} \perp \overline{CF}</math>, and <math>\overline{GK} \perp \overline{BM}</math>. Prove that <math>\overline{CF} \parallel \overline{GK}</math>.</p> <p><math>\overline{AL} \parallel \overline{BM}</math> <span style="float: right;"><i>Given</i></span></p> <p><math>\overline{AL} \perp \overline{CF}</math> <span style="float: right;"><i>Given</i></span></p> <p><math>\overline{GK} \perp \overline{BM}</math> <span style="float: right;"><i>Given</i></span></p> <p><math>m\angle ADC = 90^\circ</math> <span style="float: right;"><i>Definition of perpendicular lines</i></span></p> <p><math>m\angle BJH = 90^\circ</math> <span style="float: right;"><i>Definition of perpendicular lines</i></span></p> <p><math>m\angle BJH = m\angle AHG</math> <span style="float: right;"><i>If two parallel lines are cut by a transversal, then the corresponding angles are equal in measure</i></span></p> <p><math>m\angle ADC = m\angle AHG</math> <span style="float: right;"><i>Substitution Property of Equality</i></span></p> <p><math>\overline{CF} \parallel \overline{GK}</math> <span style="float: right;"><i>If two lines are cut by a transversal such that the corresponding angles are equal in measure, then the lines are parallel</i></span></p>	
<p>2. Given that <math>\angle B</math> and <math>\angle C</math> are supplementary and <math>m\angle A = m\angle C</math>, prove that <math>\overline{AD} \parallel \overline{BC}</math>.</p> <p><math>\angle B</math> and <math>\angle C</math> are supplementary <span style="float: right;"><i>Given</i></span></p> <p><math>m\angle A = m\angle C</math> <span style="float: right;"><i>Given</i></span></p> <p><math>m\angle B + m\angle C = 180</math> <span style="float: right;"><i>Definition of supplementary angles</i></span></p> <p><math>m\angle B + m\angle A = 180</math> <span style="float: right;"><i>Substitution Property of Equality</i></span></p> <p><math>\overline{AD} \parallel \overline{BC}</math> <span style="float: right;"><i>If a transversal intersects two lines such that the same side interior angles are supplementary, then the lines are parallel</i></span></p>	
<p>3. Mathematicians state that if a transversal to two parallel lines is perpendicular to one of the lines, then it is perpendicular to the other. Prove this statement. (Include a labeled drawing with your proof.)</p> <p><math>\overline{AL} \parallel \overline{BM}</math> <span style="float: right;"><i>Given</i></span></p> <p><math>\overline{AL} \perp \overline{CF}</math> <span style="float: right;"><i>Given</i></span></p> <p><math>m\angle ADC = 90^\circ</math> <span style="float: right;"><i>Definition of perpendicular lines</i></span></p> <p><math>m\angle ADC = m\angle BED</math> <span style="float: right;"><i>If parallel lines are cut by a transversal, then corresponding angles are equal in measure</i></span></p> <p><math>m\angle BED = 90^\circ</math> <span style="float: right;"><i>Substitution</i></span></p> <p><math>\overline{CF} \perp \overline{BM}</math> <span style="float: right;"><i>Definition of perpendicular lines</i></span></p>	

4. In the figure,  $\overline{AB} \parallel \overline{CD}$  and  $\overline{EF} \parallel \overline{GH}$ . Prove that  $m\angle AFE = m\angle DKH$ .

$\overline{AB} \parallel \overline{CD}$

Given

$\overline{EF} \parallel \overline{GH}$

Given

$m\angle AFE = m\angle GJF$

If two parallel lines are cut by a trans., then the corr. angles are equal in measure

$m\angle JKC = m\angle GJF$

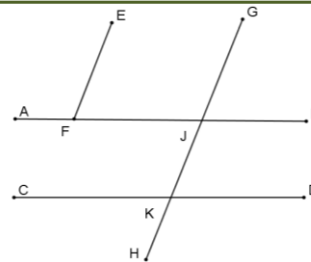
If two parallel lines are cut by a transversal, then the corresponding angles are equal in measure

$m\angle JKC = m\angle DKH$

Vertical angles are equal in measure

$m\angle AFE = m\angle DKH$

Transitive Property



5. In the figure,  $\angle E$  and  $\angle AFE$  are complementary and  $\angle C$  and  $\angle BDC$  are complementary. Prove that  $\overline{AE} \parallel \overline{CB}$ .

$\angle E$  and  $\angle AFE$  are complementary

Given

$\angle C$  and  $\angle BDC$  are complementary

Given

$m\angle E + m\angle AFE = 90^\circ$

Definition of comp. angles

$m\angle C + m\angle BDC = 90^\circ$

Definition of comp. angles

$m\angle E + m\angle AFE + m\angle A = 180^\circ$

Sum of the angle measures in a triangle is  $180^\circ$

$m\angle C + m\angle BDC + m\angle B = 180^\circ$

Sum of the angle measures in a triangle is  $180^\circ$

$m\angle A = 90^\circ; m\angle B = 90^\circ$

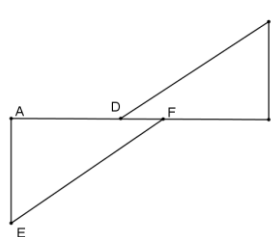
Subtraction Property of Equality

$m\angle A = m\angle B$

Substitution Property of Equality

$\overline{AE} \parallel \overline{CB}$

If two lines are cut by a transversal such that a pair of alternate interior angles are equal in measure, then the lines are parallel



6. Given a line  $l$  and a point  $P$  not on the line, the following directions can be used to draw a line  $m$  perpendicular to the line  $l$  through the point  $P$  based upon a rotation by  $180^\circ$ :
- Pick and label a point  $A$  on the line  $l$  so that the circle (center  $P$ , radius  $\overline{AP}$ ) intersects  $l$  twice.
  - Use a protractor to draw a perpendicular line  $n$  through the point  $A$  (by constructing a  $90^\circ$  angle).
  - Use the directions in Example 2 to construct a parallel line  $m$  through the point  $P$ .

Do the construction. Why is the line  $m$  perpendicular to the line  $l$  in the figure you drew? Why is the line  $m$  the only perpendicular line to  $l$  through  $P$ ?

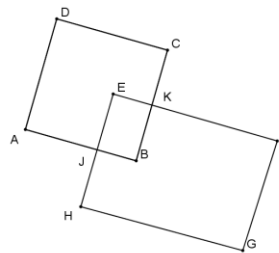
Figures will vary, but should follow directions outlined.

Student can use Homework Problem 3, or reprove the fact that if a transversal to two parallel lines is perpendicular to one then it is perpendicular to the other.

Suppose line  $m'$  contains  $P$  and is perpendicular to  $l$ , then by the interior angles theorem (or alternate interior angles or corresponding interior angles),  $m'$  is parallel to  $n$ . Since  $m$  and  $m'$  are both parallel to  $n$  and contain  $P$ , by the parallel postulate they must be the same line, i.e.,  $m' = m$ .

A student might answer, "parallel postulate." If so, partial credit might be awarded (depending on your tastes for partial credit).

Questions 7-10 all refer to the figure at the right. The exercises are otherwise unrelated to each other.



7.  $\overline{AD} \parallel \overline{BC}$  and  $\angle EJB$  is supplementary to  $\angle JBK$ . Prove that  $\overline{AD} \parallel \overline{JE}$ .

$\overline{AD} \parallel \overline{BC}$

Given

$\angle EJB$  is supplementary to  $\angle JBK$

Given

$\overline{EJ} \parallel \overline{BC}$

If a transversal intersects two lines such that the same side interior angles are supplementary, then the lines are parallel.

$\overline{AD} \parallel \overline{EJ}$

If two segments are parallel to the same segment, then they are parallel to each other.

8.  $\overline{AD} \parallel \overline{FG}$  and  $\overline{EJ} \parallel \overline{FG}$ . Prove that  $\angle DAJ$  and  $\angle EJA$  are supplementary.

$\overline{AD} \parallel \overline{FG}$

Given

$\overline{EJ} \parallel \overline{FG}$

Given

$\overline{AD} \parallel \overline{EJ}$

If two segments are parallel to the same segment, then they are parallel to each other

$m\angle DAJ + m\angle EJA = 180^\circ$

If a transversal intersects two parallel lines then the interior angles on the same side of the transversal are supplementary

9.  $m\angle C = m\angle G$  and  $\angle B$  is supplementary to  $\angle G$ . Prove that  $\overline{DC} \parallel \overline{AB}$ .

$m\angle C = m\angle G$

Given

$\angle B$  is supplementary to  $\angle G$

Given

$m\angle B + m\angle G = 180^\circ$

Definition of supplementary angles

$m\angle C + m\angle G = 180^\circ$

Substitution Property of Equality

$\overline{DC} \parallel \overline{AB}$

If a transversal intersects two lines such that the same side interior angles are supplementary, then the lines are parallel

10.  $\overline{AB} \parallel \overline{EF}$ ,  $\overline{EF} \perp \overline{CB}$ , and  $\angle EKC$  is supplementary to  $\angle KCD$ . Prove that  $\overline{AB} \parallel \overline{DC}$ .

$\overline{AB} \parallel \overline{EF}$

Given

$\overline{EF} \perp \overline{CB}$

Given

$\angle EKC$  is supplementary to  $\angle KCD$

Given

$m\angle ABC + m\angle BKE = 180^\circ$

If parallel lines are cut by a transversal, then interior angles on the same side are supplementary

$m\angle EKC + m\angle KCD = 180^\circ$

Definition of supplementary angles

$m\angle BKE = 90^\circ$  and  $m\angle EKC = 90^\circ$

Definition of right angles

$m\angle ABC = 90^\circ$  and  $m\angle KCE = 90^\circ$

Subtraction Property of Equality

$\angle ABC$  and  $\angle KCE$  are supplementary

Definition of supplementary angles

$\overline{AB} \parallel \overline{DC}$

If two lines are cut by a transversal such that a pair of interior angles on the same side are supplementary, then the lines are parallel