



Lesson 1: Multiplying and Factoring Polynomial Expressions

Student Outcomes

- Students use the distributive property to multiply a monomial by a polynomial and understand that factoring reverses the multiplication process.
- Students use polynomial expressions as side lengths of polygons and find area by multiplying.
- Students recognize patterns and formulate shortcuts for writing the expanded form of binomials whose expanded form is a perfect square or the difference of perfect squares.

Lesson Notes

Central to the concepts in this lesson is A-APR.1 and understanding the system and operations of polynomial expressions, specifically multiplication and factoring of polynomials. Lengths of time suggested for the examples and exercises in this lesson assume that students remember some of what is presented in the examples from work in earlier modules and earlier grades. Students may need more or less time with this lesson than is suggested. The teacher should make decisions about how much time is needed for these concepts based on the needs of their students.

MP.4

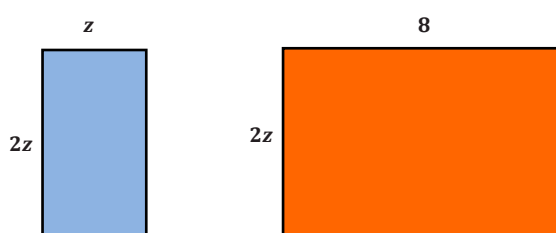
This lesson asks students to use geometric models to demonstrate their understanding of multiplication of polynomials.

Classwork

Opening Exercise (4 minutes)

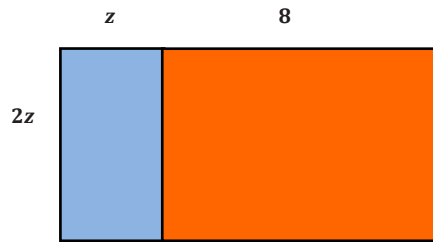
Opening Exercise

Write expressions for the areas of the two rectangles in the figures given below.



Blue: $2z(z) = 2z^2$; Orange: $2z(8) = 16z$

Now write an expression for the area of this rectangle:



$$2z^2 + 16z$$

- How did you find your answer for the second rectangle?
 - Add the two areas in part (a) or, multiply the length of the rectangle by the width $2z(z + 8) = 2z^2 + 16z$. Both give the same final result.
- If you find the area by multiplying the total length times total width what property of operations are you using?
 - The distributive property.
- What would be another way to find the total area?
 - Finding the area of the two separate rectangles and adding their area: $2z(z) + 2z(8) = 2z^2 + 16z$

Scaffolding:

Model the following examples of polynomial multiplication for students who need to review:

Multiply two monomials: (Point out to students that because multiplication is both commutative and associative, factors may be reordered to group the numerical factors together and then the variable factors together.)

$$5ab \cdot 4c = (5 \cdot 4)(ab \cdot c) = 20abc$$

$$3x^2 \cdot 4x^3y \cdot 5y^2 = (3 \cdot 4 \cdot 5)(x^2 \cdot x^3 \cdot y^1 \cdot y^2) = 60x^{2+3}y^{1+2} = 60x^5y^3$$

Multiply a polynomial by a monomial: Some students may benefit from relating multiplication of polynomials to multiplication of numbers in base 10. In the example below, the multiplication process is represented vertically (like a base-10 product of a 2-digit by a 1-digit numbers) and then horizontally, using the distributive property.

Multiply $(5a + 7b)$ by $3c$. To find this product vertically, follow the same procedure as you would with place values for whole numbers. Just be sure to follow the rules for combining like terms. Show how to multiply vertically:

$$\begin{array}{r} 5a + 7b \\ \times \quad 3c \\ \hline 15ac + 21bc \end{array}$$

Now, multiply the polynomial by the monomial horizontally, using the distributive property for multiplication over addition. Make sure each term of the first binomial is distributed over both terms of the second.

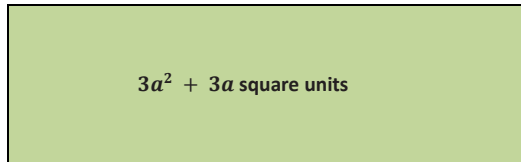
$$(5a + 7b)3c = (5a \cdot 3c) + (7b \cdot 3c) = 15ac + 21bc \text{ (The Associative Property for Multiplication allows us to group the numbers and the variables together.)}$$

Example 1 (3 minutes)

Have students work on this example with a partner or in small groups. Have the groups share their process and their findings and discuss the differences in processes used (if there are any).

Example 1

The total area of this rectangle is represented by $3a^2 + 3a$. Find expressions for the dimensions of the total rectangle.



$$3a^2 + 3a = 3a(a + 1) \text{ [Students may opt to factor only } a \text{ or } 3: a(3a + 3) \text{ or } 3(a^2 + a)]$$

In the example above, _____ and _____ are factors of $3a^2 + 3a$ and could represent the dimensions of the rectangle.

$3a$ and $(a + 1)$, or 3 and $(a^2 + a)$, or a and $(3a + 3)$

Note: If students try to use 1 as the common factor for two or more numbers, point out that, while 1 is indeed a factor, factoring out a 1 does not help in finding the factors of an expression. If this issue arises, it may be necessary to discuss the results when factoring out a 1.

Factoring out the Greatest Common Factor (GCF)

Students now revisit factoring out the greatest common factor as was introduced in Grade 6, Module 2.

- When factoring a polynomial, we first look for a monomial that is the greatest common factor (GCF) of all the terms of the polynomial. Then, we reverse the distribution process by factoring the GCF out of each term and writing it on the outside of the parentheses.
- In the example above we factored out the GCF: $3a$.

Exercises 1–3 (3 minutes)

For the exercises below have students work with a partner or in small groups to factor out the GCF for each expression.

Exercises 1–3

Factor each by factoring out the Greatest Common Factor:

1. $10ab + 5a$

$5a(2b + 1)$

2. $3g^3h - 9g^2h + 12h$

$3h(g^3 - 3g^2 + 4)$

3. $6x^2y^3 + 9xy^4 + 18y^5$

$3y^3(2x^2 + 3xy + 6y^2)$ [Students may find this one to be more difficult. It is used as an example in a scaffolding suggestion below.]

Discussion: The Language of Polynomials (4 minutes)

Make sure the students have a clear understanding of the following terms and use them appropriately during instruction. The scaffolding suggestion below may be used to help students understand the process of factoring out the GCF. Begin the discussion by reviewing the definition of prime and composite numbers given in the student materials.

Discussion: The Language of Polynomials

A **prime number** is a positive integer greater than 1 whose only positive integer factors are 1 and itself.

A **composite number** is a positive integer greater than 1 that is not a prime number.

A composite number can be written as the product of positive integers with at least one factor that is not 1 or itself.

For example, the prime number 7 has only 1 and 7 as its factors. The composite number 6 has factors of 1, 2, 3, and 6; it could be written as the product $2 \cdot 3$.

- Factoring is the reverse process of multiplication (through multiple use of the distributive property). We factor a polynomial by reversing the distribution process—factoring the GCF out of each term and writing it on the outside of the parentheses. To check whether the polynomial's factored form is equivalent to its expanded form, you can multiply the factors to see if the product yields the original polynomial.
- $4(x + 3)$ is called a *factored form* of $4x + 12$.

A nonzero polynomial expression with integer coefficients is said to be prime or irreducible over the integers if it satisfies two conditions:

- it is not equivalent to 1 or -1 , and
- if the polynomial is written as a product of two polynomial factors, each with integer coefficients, then one of the two factors must be 1 or -1 .

- Note: this definition actually specifies prime numbers and their negatives as well (the case when the polynomial has degree 0).
- For example: $4x + 9$ is irreducible over the integers.

Given a polynomial in standard form with integer coefficients, let c be the greatest common factor of all of the coefficients. The polynomial is *factored completely over the integers* when it is written as a product of c and one or more prime polynomial factors, each with integer coefficients.

- In the future, we will learn to factor over the rationals and reals.

Scaffolding:

For students who struggle with factoring the GCF from a more complicated polynomial, suggest they use a chart to organize the terms and factors. Here is an example using Exercise 3 above:

Stack the three terms (monomial expressions) on the far left of a table and then write each of the terms of the polynomial in prime factor form across the row, stacking those that are the same. Then shade or circle the columns that have the same factor for all three terms:

$6x^2y^3$	2	3		x	x	y	y	y		
$9xy^4$	3	3		x		y	y	y	y	
$18y^5$	2	3	3			y	y	y	y	y
		3				y	y	y		

$2x^2$
 $3xy$
 $6y^2$

}

→

GCF: $3y^3$

Now, look down the columns to find which factors are in all three rows. The blue columns show those common factors, which are shared by all three terms. So, the greatest common factor (GCF) for the three terms is the product of those common factors: $3y^3$. This term is written on the outside of the parentheses, then reversing the distributive property, we write the remaining factors inside the parentheses for each of the terms that are not in the blue shaded columns. (It may be helpful to point out that factoring out the GCF is the same as dividing each term by the GCF.) In this example, it is $3y^3(2x^2 + 3xy + 6y^2)$. You can find the GCF by multiplying the factors across the bottom and the terms of the other factor by multiplying the remaining factors across each row.

Example 2 (4 minutes): Multiply Two Binomials

Demonstrate that the product can be found by applying the distributive property (twice) where the first binomial distributes over each of the second binomial’s terms and relate the result to the area model as was used in Module 1 and as shown below. Note that while the order of the partial products shown corresponds with the well-known “FOIL” method (Firsts, Outs, Inners, Lasts), teachers are discouraged from teaching polynomial multiplication as a procedure or with mnemonic devices such as “FOIL”. Instead, foster understanding by relating the process to the distributive property and the area model.

Since side lengths of rectangles cannot be negative, it is not directly applicable to use the area model for multiplying general polynomials. (We cannot be certain that each term of the polynomial represents a positive quantity.) However, we can use a tabular method that resembles the area model to track each partial product as we use the distributive property to multiply the polynomials.

Example 2: Multiply Two Binomials

Using a Table as an Aid

You have seen the geometric area model used in previous examples to demonstrate the multiplication of polynomial expressions for which each expression was known to represent a measurement of length.

Without a context such as length, we cannot be certain that a polynomial expression represents a positive quantity.

Therefore, an area model is not directly applicable to all polynomial multiplication problems. However, a table can be used in a similar fashion to identify each partial product as we multiply polynomial expressions. The table serves to remind us of the area model even though it does not represent area.

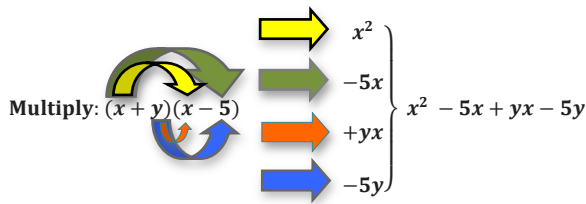
For example, fill in the table to identify the partial products of $(x + 2)(x + 5)$. Then, write the product of $(x + 2)(x + 5)$ in standard form.

	x	$+$	5	
x	x^2		$5x$	
$+$				
2	$2x$		10	

$x^2 + 7x + 10$

Without the Aid of a Table

Regardless of whether or not we make use of a table as an aid, the multiplying of two binomials is an application of the distributive property. Both terms of the first binomial distribute over the second binomial. Try it with $(x + y)(x - 5)$. In the example below, the colored arrows match each step of the distribution with the resulting partial product:



Example 3 (4 minutes): The Difference of Squares

Example 3: The Difference of Squares

Find the product of $(x + 2)(x - 2)$. Use the distributive property to distribute the first binomial over the second.

With the Use of a Table:

	x	$+$	2	
x	x^2		$2x$	
$+$				
-2	$-2x$		4	

$x^2 - 4$

Without the Use of a Table:

$$(x)(x) + (x)(-2) + (2)(x) + (+2)(-2) = x^2 - 2x + 2x - 4 = x^2 - 4$$

- Do you think the linear terms will always be opposites when we multiply the sum and difference of the same two terms? Why?
 - Yes, because when we multiply the first term of the first binomial by the last term of the second we get the opposite of what we get when we multiply the second term of the first binomial by the first term of the second.
- So, $x^2 - 4$ is the difference of two perfect squares. Factoring the difference of two perfect squares reverses the process of finding the product of the sum and difference of two terms.

Exercise 4 (6 minutes)

The following can be used as a guided practice or as independent practice.

Exercise 4

Factor the following examples of the difference of perfect squares.

- | | | |
|----|-----------------|----------------------|
| a. | $t^2 - 25$ | $(t - 5)(t + 5)$ |
| b. | $4x^2 - 9$ | $(2x - 3)(2x + 3)$ |
| c. | $16h^2 - 36k^2$ | $(4h - 6k)(4h + 6k)$ |
| d. | $4 - b^2$ | $(2 - b)(2 + b)$ |
| e. | $x^4 - 4$ | $(x^2 - 2)(x^2 + 2)$ |
| f. | $x^6 - 25$ | $(x^3 - 5)(x^3 + 5)$ |

Point out that any even power can be a perfect square and that 1 is always a square.

Write a General Rule for Finding the Difference of Squares

Write $a^2 - b^2$ in factored form.

$$(a + b)(a - b)$$

Exercises 5–7 (4 minutes)

The following exercises may be guided or modeled, depending on how well students did on the previous example.

Exercises 5–7

For each of the following differences of squares, factor completely over the integers.

5. $9y^2 - 100z^2$
 $(3y + 10z)(3y - 10z)$
6. $a^4 - b^6$
 $(a^2 + b^3)(a^2 - b^3)$
7. $r^4 - 16s^4$ (Hint: This one will factor twice.)
 $(r^2 + 4s^2)(r^2 - 4s^2) = (r^2 + 4s^2)(r - 2s)(r + 2s)$

Example 4 (4 minutes): The Square of a Binomial

It may be worthwhile to let students try their hands at finding the product before you show them how. If students struggle to include every step in the process, pause at each step so that they have time to absorb the operations that took place.

Example 4: The Square of a Binomial

To square a binomial, such as $(x + 3)^2$, multiply the binomial by itself.

$$(x + 3)(x + 3) = (x)(x) + (x)(3) + (3)(x) + (3)(3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9$$

Square the following general examples to determine the general rule for squaring a binomial:

a. $(a + b)^2$

$$(a + b)(a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

b. $(a - b)^2$

$$(a - b)(a - b) = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2$$

Point out that the process used in squaring the binomial is called **expanding**; in general, *expanding* means rewriting a product of sums as a sum of products through use of the distributive property.

- How are the answers to the two general examples similar? How are they different? What is the cause of the difference between the two?
 - Both results are quadratic expressions with three terms. The first and second examples both have an a^2 , b^2 , and an ab term. In Example 2, the ab term is negative, while it is positive in Example 1. The negative (subtraction) in the binomial's constant term in Example 2 causes the middle term to be negative.

Exercises 8–9 (3 minutes)**Exercises 8–9**

Square the binomial.

8. $(a + 6)^2$

$$a^2 + 12a + 36$$

9. $(5 - w)^2$

$$25 - 10w + w^2$$

Closing (2 minutes)

- Factoring is the reverse process of multiplication.
- Look for a GCF first when you are factoring a polynomial.
- Keep factoring until all the factors are prime.
- Factor the difference of squares $a^2 - b^2$ as $(a - b)(a + b)$
- Factor $a^2 + 2ab + b^2$ as $(a + b)^2$.
- Factor $a^2 - 2ab + b^2$ as $(a - b)^2$.

Lesson Summary

Factoring is the reverse process of multiplication. When factoring, it is always helpful to look for a GCF that can be pulled out of the polynomial expression. For example, $3ab - 6a$ can be factored as $3a(b - 2)$.

Factor the difference of perfect squares $a^2 - b^2$

$$\text{as } (a - b)(a + b).$$

When squaring a binomial $(a + b)$

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Exit Ticket (4 minutes)

Name _____

Date _____

Lesson 1: Multiplying and Factoring Polynomial Expressions

Exit Ticket

When you multiply two terms by two terms, you should get four terms. Why is the final result when you multiply two binomials sometimes only three terms? Give an example of how your final result can end up with only two terms.

Exit Ticket Sample Solutions

When you multiply two terms by two terms, you get four terms. Why does the final result when you multiply two binomials sometimes come out with three terms? Give an example of how your final result can be two terms.

Often when you multiply two binomials, each has a term with the same variable, say x , and two of the terms combine to make one single x -term. If the two terms combine to make zero there will be only two of the four terms left. For example, $(x + 3)(x - 3) = x^2 - 9$.

Problem Set Sample Solutions

1. For each of the following factor out the greatest common factor:

a. $6y^2 + 18$

$$6(y^2 + 3)$$

b. $27y^2 + 18y$

$$9y(3y + 2)$$

c. $21b - 15a$

$$3(7b - 5a)$$

d. $14c^2 + 2c$

$$2c(7c + 1)$$

e. $3x^2 - 27$

$$3(x^2 - 9)$$

2. Multiply:

a. $(n - 5)(n + 5)$

$$n^2 - 25$$

b. $(4 - y)(4 + y)$

$$16 - y^2$$

c. $(k + 10)^2$

$$k^2 + 20k + 100$$

d. $(4 + b)^2$

$$16 + 8b + b^2$$

3. The measure of a side of a square is x units. A new square is formed with each side 6 units longer than the original square's side. Write an expression to represent the area of the new square. (Hint: Draw the new square and count the squares and rectangles.)

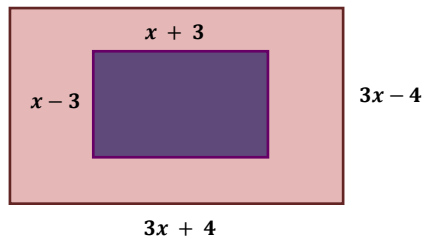
Original square x



New square:

$$(x + 6)^2 \text{ or } x^2 + 12x + 36$$

4. In the accompanying diagram, the width of the inner rectangle is represented by $x - 3$ and its length by $x + 3$. The width of the outer rectangle is represented by $3x + 4$ and its length by $3x - 4$.



- a. Write an expression to represent the area of the larger rectangle.

Find the area of the larger (outer) rectangle by multiplying the binomials:

$$(3x - 4)(3x + 4) = 9x^2 - 16$$

- b. Write an expression to represent the area of the smaller rectangle.

Find the area of the smaller (inner) rectangle by multiplying:

$$(x - 3)(x + 3) = x^2 - 9$$

- c. Express the area of the pink shaded region as a polynomial in terms of x . (Hint: You will have to add or subtract polynomials to get your final answer.)

Subtract the area of the smaller from the area of the larger rectangle:

$$(9x^2 - 16) - (x^2 - 9) = 9x^2 - 16 - x^2 + 9 = 8x^2 - 7, \text{ which is the area of the pink shaded region.}$$