Lesson 17: Graphing Quadratic Functions from the Standard Form, $f(x) = ax^2 + bx + c$

Student Outcomes

- Students graph a variety of quadratic functions using the form $f(x) = ax^2 + bx + c$ (standard form).
- Students analyze and draw conclusions about contextual applications using the key features of a function and its graph.

MP.2 Throughout this lesson, students are presented with a verbal description where a relationship can be modeled symbolically and graphically. Students must de-contextualize the verbal description and graph the quadratic relationship they describe. They then contextualize their solution to answer the questions posed by the examples fully.

Lesson Notes

Students may wonder why all physics applications of quadratic functions have the same leading coefficients. Lesson 23 has a teaching moment for students to learn the laws of physical objects in motion. Here is a simplified way to explain it if they have questions now:

Due to Earth's gravity, there is a constant downward force on any object. The force is proportional to the mass of the object. The proportional constant is called the gravitational acceleration. The constant is -32 *ft/sec²*, or -9.8 *m/sec²*, near the Earth's surface.

- When we use a quadratic function to model the height (or vertical position) over time of any falling or projected object, the leading coefficient is calculated to be half of the gravitational acceleration. Therefore, the leading coefficient must either be -16 or -4.9, depending on the unit of choice.
- The coefficient of the linear term in the function represents the initial vertical speed of the object (if in feet, then feet per second or if in meters, then meters per second).
- The constant of the quadratic function represents the initial height (or vertical position) of the object.

In summary, the following quadratic function $h(t)$ can be used to describe the height as a function of time for any projectile motion under a constant vertical acceleration due to gravity.

$$h(t) = \frac{1}{2}gt^2 + v_{0t} + h_0$$

t: time (in sec)

g: gravitational acceleration (in ft./sec^2 or m/sec^2)

v_0: initial vertical speed (in ft./sec. or m/sec.)

h_0: initial height (in ft. or m)

Classwork

Opening Exercise (10 minutes)

Present the students with the following problem. Write or project it on the board or screen and have students work with a partner or in small groups to solve the problem.

Opening Exercise

A high school baseball player throws a ball straight up into the air for his math class. The math class was able to determine that the relationship between the height of the ball and the time since it was thrown could be modeled by the function, $h(t) = -16t^2 + 96t + 6$, where t represents the time (in seconds) since the ball was thrown and h, the height (in feet) of the ball above the ground.

Scaffolding:

You may want to start by substituting values and creating a graph together as a class before getting into the problem analysis. Students can then use that graph as a visual reference during the discussion.

Have students informally consider and discuss the following questions in a small group or with a partner. Some possible student responses are provided after each question.

a. **What does the domain of the function represent in this context?**

The time (number of seconds) since the ball was thrown.

b. **What does the range of this function represent?**

The height (in feet) of the ball above the ground.

c. **At what height does the ball get thrown?**

The initial height of the ball is when t is 0 sec. (i.e., $h(0)$), which is the y-intercept. The initial height is 6 ft.

d. **After how many seconds does the ball hit the ground?**

The ball's height is 0 when $h(t) = 0$. We can solve using any method, since this does not appear to be easily factorable and the size of the numbers might be cumbersome in the quadratic formula, let's solve by completing the square:

$$-16t^2 + 96t + 6 = 0$$
$$-16(t^2 - 6t) = -6$$
$$-16(t^2 - 6t + 9) = -6 - 144$$

from here we see the completed-square form: $h(t) = -16(t-3)^2 + 150$

$$-16(t-3)^2 = -150$$
$$(t-3)^2 = \frac{150}{16}$$
$$t - 3 = \pm\frac{\sqrt{150}}{4}$$
$$t = 3 \pm \frac{\sqrt{150}}{4}$$
$$t \approx 6.0618 \ldots or - 0.0618 \ldots$$

For this context, the ball hits the ground at approximately 6.2 sec.

e. **What is the maximum height that the ball reaches while in the air? How long will the ball take to reach its maximum height?**

Completing the square (and using the work from the previous question) we get $h(t) = -16(t-3)^2 + 150$, so the vertex is $(3, 150)$ meaning that the maximum height is 150 ft. and it will reach that height in 3 sec.

f. **What feature(s) of this quadratic function are "visible" since it is presented in the standard form, $f(x) = ax^2 + bx + c$?**

We can see the initial position, or height of the ball, or the height when $t = 0$, in the constant term. We can also see the leading coefficient, which tells us about the end behavior and whether the graph is wider or narrower than the graph of $f(x) = x^2$.

g. **What feature(s) of this quadratic function are "visible" when it is rewritten in vertex form, $f(x) = a(x - h)^2 + k$?**

We can only see the coordinates of the vertex and know that $x = h$ is the equation of the axis of symmetry. We can still see the leading coefficient in this form, which tells us about the end behavior and whether the graph is wider or narrower than the graph of $f(x) = x^2$.

To understand and solve a problem presented in a context, point out the importance of graphing the function, along with interpreting the domain and range. Demonstrate how the key features of a quadratic function that are discoverable from the algebraic form can help us create the graph. We will use the function from the Opening Exercise as our first example.

Have students contemplate and write a general strategy for graphing a quadratic function from the standard form in their student materials. Discuss or circulate to ensure what they have written will be helpful to them later. Note that a correct strategy is provided in the lesson summary.

A general strategy for graphing a quadratic function from the standard form:

- *Look for hints in the function's equation for general shape, direction, and y-intercept.*
- *Solve $f(x) = 0$ to find the x-intercepts by factoring, completing the square, or using the quadratic formula.*
- *Find the vertex by completing the square or using symmetry. Find the axis of symmetry and the x-coordinate of the vertex using $\frac{-b}{2a}$ and the y-coordinate of the vertex by finding $f\left(\frac{-b}{2a}\right)$.*
- *Plot the points you know (at least three are required for a unique quadratic function), sketch the graph of the curve that connects them, and identify the key features of the graph.*

Example 1 (10 minutes)

Have students use the steps to graph the baseball throw example in the Opening Exercise: $h(t) = -16t^2 + 96t + 6$. Have students answer the following questions in their student materials with a partner or in small groups.

Example 1

A high school baseball player throws a ball straight up into the air for his math class. The math class was able to determine that the relationship between the height of the ball and the time since it was thrown could be modeled by the function, $h(t) = -16t^2 + 96t + 6$, where t represents the time (in seconds) since the ball was thrown and h, the height (in feet) of the ball above the ground.

Remind students to look back at the work done in the Opening Exercise.

a. **What do you notice about the equation, just as it is, that will help us in creating our graph?**

The leading coefficient is negative, so we know the graph opens down. $h(0) = 6$, so the point $(0, 6)$ is on the graph.

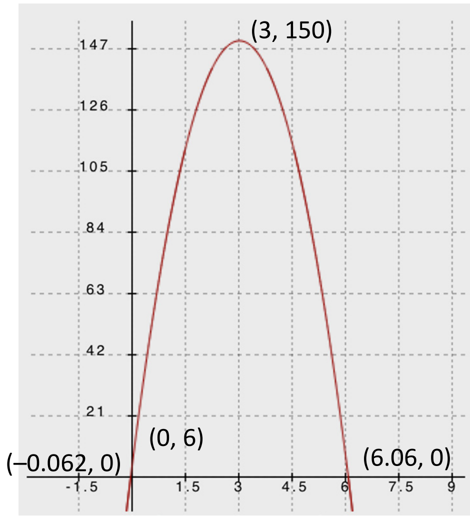
b. **Can we factor to find the zeros of the function? If not, solve $h(t) = 0$ by completing the square.**

This function is not factorable, so we complete the square to find the zeros to be $(6.06, 0)$ *and* $(-0.062, 0)$.

c. **Which will you use to find the vertex? Symmetry? Or the completed-square form of the equation?**

Since we already completed the square (and the zeros are irrational, so more difficult to work with), we can easily find the vertex as $(3, 150)$.

d. **Now we plot the graph of $h(t) = -16t^2 + 96t + 6$ and identify the key features in the graph.**



After students have graphed the function, ask the following questions to probe further into their conceptual understanding of the graphic representation:

- What is the appropriate domain for the context of the function?
  - *Since the time must be positive, the domain for the context is* $[0, 6.06]$.
- What do the 3 and the 150 in the vertex tell us?
  - *That the ball reached its highest point of* 150 *ft. after* 3 *sec., and then it started back down.*
- What do the 6.06 and -0.062 of the zeros tell us about the ball's flight?
  - *Since the zeros tell us when the ball was at ground level (height* $= 0$*), the negative value indicates that the ball was above the ground at the time of the throw*. The* 6.06 *tells us that it took* 6.06 *sec. for the ball to complete its flight and hit the ground.*

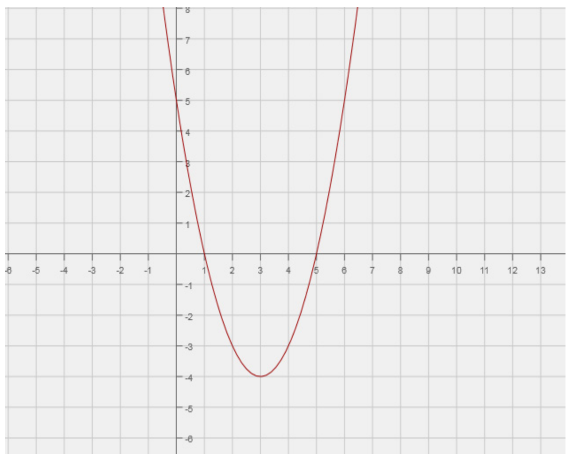
    **The* -0.062 *does not describe part of the baseball throw. It is a thought experiment using mathematics. Hypothetically, the* -0.062 *could mean, based on the graph, that if we backtracked in time and asked, "What if instead of starting at* 6 *ft. high at time* 0*, we assume the ball was somehow thrown up from the ground level at an earlier time, and reached* 6 *ft and time* 0*? Then,* -0.062 *would be the time the ball got started?"*
- Does this curve represent the path of the ball? Explain.
  - *No, the problem says that the ball went straight up so it would probably come straight down. The graph does not show forward movement for the ball but only represents the elapsed time as it relates to the ball's height.*

Exercises 1–5 (20 minutes)

Students use the steps in the examples above to complete the following exercises. The first two are pure mathematical examples. Then, the next two will be in a context.

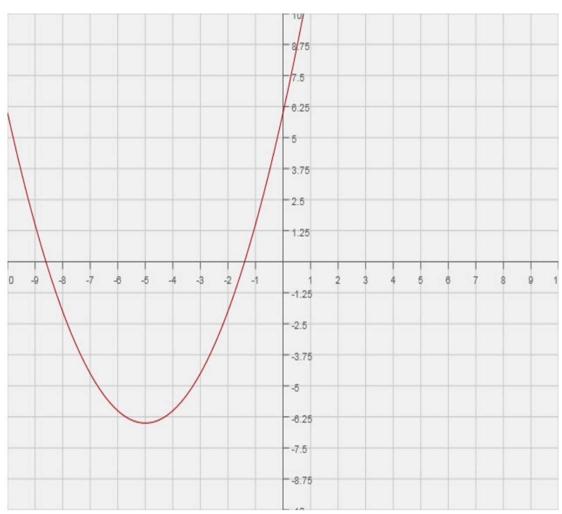
Exercises 1–5

1. **Graph the equation $n(x) = x^2 - 6x + 5$ and identify the key features.**



x-intercepts: $(5, 0)$ $(1, 0)$
y-intercept: $(0, 5)$
vertex: $(3, -4)$

2. **Graph the equation $f(x) = \frac{1}{2}x^2 + 5x + 6$ and identify the key features.**



x-intercepts: $\left(-5 + \sqrt{13}, 0\right)$ $\left(-5 - \sqrt{13}, 0\right)$
y-intercept: $(0, 6)$
vertex: $(-5, -6.5)$

3. Paige wants to start a summer lawn-mowing business. She comes up with the following profit function that relates the total profit to the rate she charges for a lawn-mowing job:

$$P(x) = -x^2 + 40x - 100$$

Both profit and her rate are measured in dollars. Graph the function in order to answer the following questions.

a. Graph $P(x)$

x-Intercepts: $(20 + 10\sqrt{3}, 0)$, $(20 - 10\sqrt{3})$
y-intercept: $(0, -100)$
vertex: $(20, 300)$

b. According to the function, what is her initial cost (e.g., maintaining the mower, buying gas, advertising)? Explain your answer in the context of this problem.

When Paige has not mown any lawns or charged anything to cut grass, her profit would be -100. A negative profit means that Paige is spending 100 dollars to run her business.

c. Between what two prices does she have to charge to make a profit?

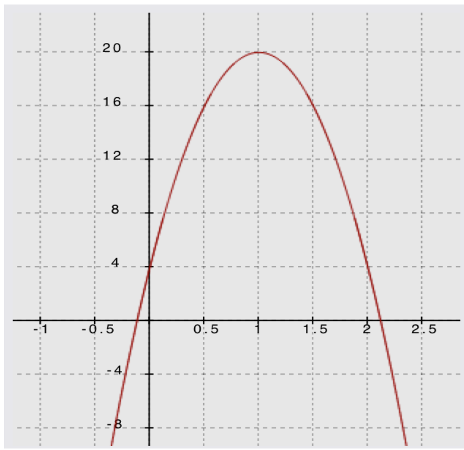
Using completing the square we find the intercepts at $(20 + 10\sqrt{3})$ and $(20 - 10\sqrt{3})$. However, since this question is about money, we approximate and find that her rates should be between $\$2.68$ and $\$37.32$.

d. If she wants to make a $\$275$ profit this summer is this the right business choice?

Yes. Looking at the graph, the vertex, or $(20, 300)$, is the maximum profit. So, if Paige charges $\$20$ for each lawn she mows she can make a $\$300$ profit, which is $\$25$ more than what she wants.

4. A student throws a bag of chips to her friend. Unfortunately, her friend does not catch the chips and the bag hits the ground. The distance from ground (height) for the bag of chips is modeled by the function $h(t) = -16t^2 + 32t + 4$, where $h(t)$ is the height (distance from the ground in feet) of the chips and t is the number of seconds the chips are in the air.

a. Graph $h(t)$.



t-intercepts: $\left(1+\frac{\sqrt{5}}{2}, 0\right)$ $\left(1-\frac{\sqrt{5}}{2}\right)$
h-intercept: $(0, 4)$
vertex: $(1, 20)$

b. From what height are the chips being thrown? Tell how you know.

4 ft. This is the initial height, or when $t = 0$.

c. What is the maximum height the bag of chips reaches while airborne? Tell how you know.

From the graph, the vertex is $(1, 20)$, which means that at 1 second the bag is 20 ft. above the ground for this problem. Since this is the vertex of the graph, and the leading coefficient of the quadratic function is negative, the graph opens down (as $t \to \pm\infty$, $h(t) \to -\infty$) and the vertex is the maximum of the function. This means 20 ft. is the maximum height of the thrown bag.

d. How many seconds after the bag was thrown did it hit the ground?

By completing the square we find that $t = 1 \pm \frac{\sqrt{5}}{2}$. Since this is time in seconds, we need a positive value: $1 + \frac{\sqrt{5}}{2}$, which is about 2.12 sec.

e. What is the average rate of change of height for the interval from 0 to $\frac{1}{2}$ second? What does that number represent in terms of the context?

$\dfrac{\left[f\left(\frac{1}{2}\right) - f(0)\right]}{\frac{1}{2} - 0} = \dfrac{[16-4]}{\frac{1}{2}} = 24$ *ft./sec, which represents the average speed of the bag of chips from 0 to $\frac{1}{2}$ sec.*

f. Based on your answer to part (e), what is the average rate of change for the interval from 1.5 to 2 sec.?

The average rate of change for the interval from 1.5 to 2 sec. will be the same as it is from 0 to $\frac{1}{2}$ except that it will be negative: -24 ft./sec.

5. **Notice how the profit and height functions both have negative leading coefficients? Explain why this is.**

The nature of both of these contexts is that they have continually changing rates and both require the graph to open down, since each would have a maximum. Problems that involve projectile motion have maximums because an object can only go so high before gravity pulls it back down. Profits also tend to increase as prices increase only to a point before sales drop off, and profits begin to fall.

Closing (2 minutes)

- For a profit function in the standard form, $P(x) = ax^2 + bx + c$, what does the constant, c, identify?
  - *Starting cost*
- For a height function in the standard form, $h(t) = at^2 + bt + c$, what does the constant, c, identify?
  - *Starting height*
- Describe a strategy for graphing a function represented in standard form, $f(x) = ax^2 + bx + c$?
  - *Examine the form of the equation for hints.*
  - *Find the zeros by factoring or completing the square.*
  - *Find the vertex by completing the square or using symmetry.*
  - *Plot the points that you know (at least three), sketch the curve, and identify the key features.*

Lesson Summary

The standard form of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$. A general strategy to graphing a quadratic function from the standard form is:

- **Look for hints in the function's equation for general shape, direction, and y-intercept.**
- **Solve $f(x) = 0$ to find the x-intercepts by factoring, completing the square, or using the quadratic formula.**
- **Find the vertex by completing the square or using symmetry. Find the axis of symmetry and the x-coordinate of the vertex using $\frac{-b}{2a}$ and the y-coordinate of the vertex by finding $f\left(\frac{-b}{2a}\right)$.**
- **Plot the points that you know (at least three are required for a unique quadratic function), sketch the graph of the curve that connects them, and identify the key features of the graph.**

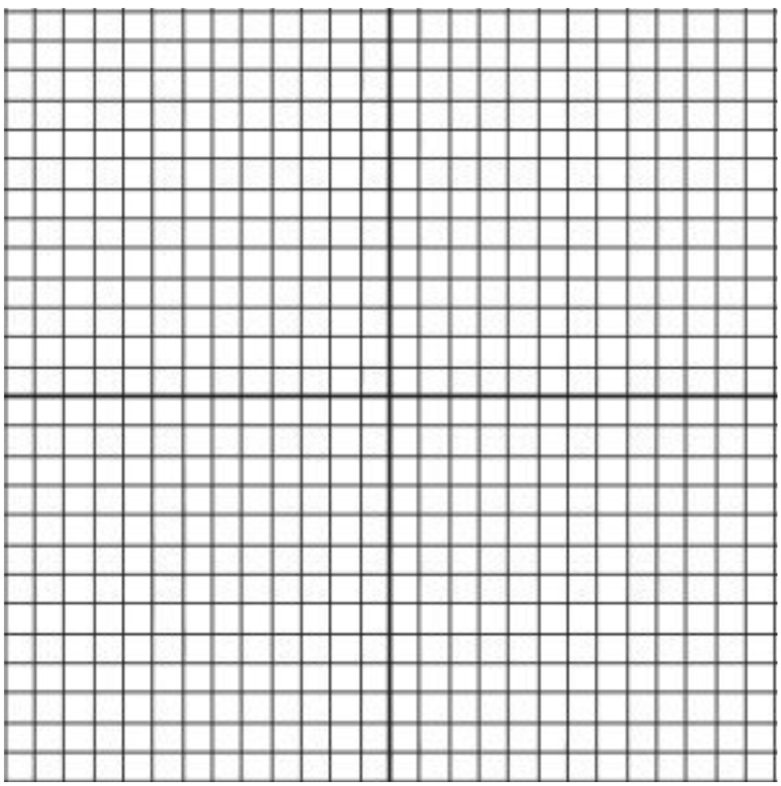
Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 17: Graphing Quadratic Functions from the Standard Form, $f(x) = ax^2 + bx + c$

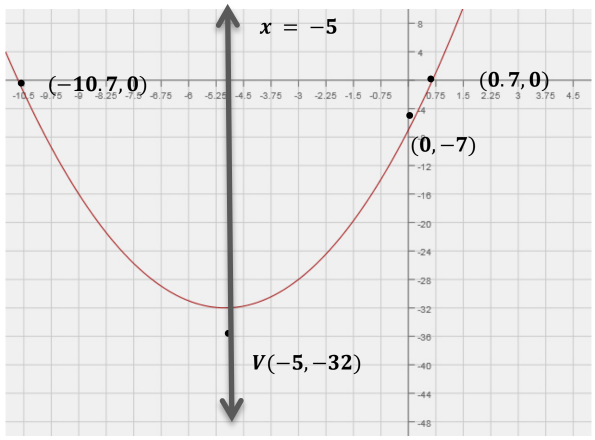
Exit Ticket

Graph $g(x) = x^2 + 10x - 7$ and identify the key features (e.g., vertex, axis of symmetry, x- and y-intercepts).



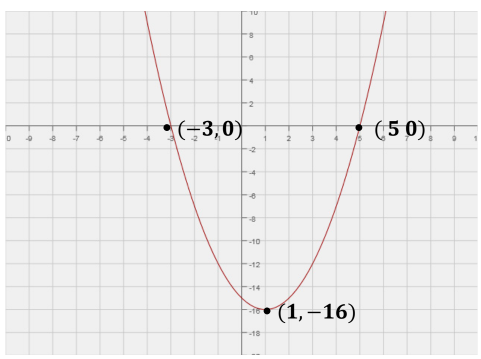
Exit Ticket Solutions

Graph $g(x) = x^2 + 10x - 7$ and identify the key features (e.g., vertex, axis of symmetry, x- and y-intercepts).

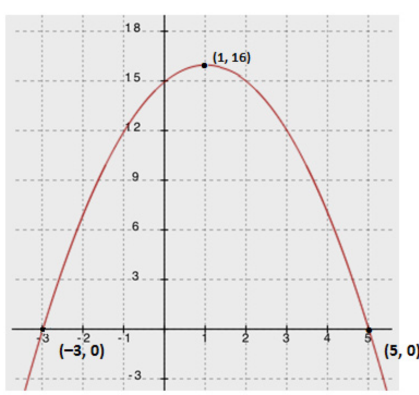


Problem Set Solutions

1. Graph $f(x) = x^2 - 2x - 15$ and identify its key features.



2. Graph the following equation $f(x) = -x^2 + 2x + 15$ and identify its key features.



3. Did you recognize the numbers in the first two problems? The equation in the second problem is the product of -1 and the first equation. What effect did multiplying the equation by -1 have on the graph?

   The graph gets reflected across the x-axis. The x-intercepts remain the same. The y-intercept becomes the opposite of the original y-intercept. The end behavior of the graph reversed. The vertex became the maximum instead of the minimum.

4. Giselle wants to run a tutoring program over the summer. She comes up with the following profit function:

   $$P(x) = -2x^2 + 100x - 25.$$

   Between what two prices should she charge to make a profit? How much should she charge her students if she wants to make the most profit?

   Using the quadratic formula the two roots are $25 \pm \frac{35\sqrt{2}}{2}$, which is about 0.25 and 50 dollars. She can't charge negative dollars, so between $\$1$ and $\$50$ Giselle can expect to make a profit. If she charges $\$25$ she will make a maximum profit of $\$1,225$.

5. Doug wants to start a physical therapy practice. His financial advisor comes up with the following profit function for his business: $P(x) = -\frac{1}{2}x^2 + 150x - 10,000$. How much will it cost for him to start the business? What should he charge his clients to make the most profit?

   The formula suggests it would cost him $\$10,000$ to start his business. He should charge $\$150$ dollars to make the most profit.