Lesson 5: The Power of Exponential Growth

Student Outcomes

- Students are able to model with and solve problems involving exponential formulas.

Lesson Notes

The primary goals of the lesson are to explore the connection between exponential growth and geometric sequences and to compare linear growth to exponential growth in context. In one exercise, students graph both types of sequences on one graph to help visualize the growth in comparison to each other. In the Closing, students are challenged to recognize that depending on the value of the base in the exponential expression of a geometric sequence, it can take some time before the geometric sequence will exceed the arithmetic sequence. Students begin to make connections in this lesson between geometric sequences and exponential growth and between arithmetic sequences and linear growth. These connections are formalized in later lessons.

Classwork

Opening Exercise (5 minutes)

Direct students to begin the lesson with the following comparison of two options.

Opening Exercise

Two equipment rental companies have different penalty policies for returning a piece of equipment late:

Company 1: On day 1, the penalty is $\$5$. On day 2, the penalty is $\$10$. On day 3, the penalty is $\$15$. On day 4, the penalty is $\$20$ and so on, increasing by $\$5$ each day the equipment is late.

Company 2: On day 1, the penalty is $\$0.01$. On day 2, the penalty is $\$0.02$. On day 3, the penalty is $\$0.04$. On day 4, the penalty is $\$0.08$ and so on, doubling in amount each additional day late.

Jim rented a digger from Company 2 because he thought it had the better late return policy. The job he was doing with the digger took longer than he expected, but it did not concern him because the late penalty seemed so reasonable. When he returned the digger 15 days late, he was shocked by the penalty fee. What did he pay, and what would he have paid if he had used Company 1 instead?

Company 1	
Day	Penalty
1	$\$5$
2	$\$10$
3	$\$15$
4	$\$20$
5	$\$25$
6	$\$30$
7	$\$35$
8	$\$40$
9	$\$45$
10	$\$50$
11	$\$55$
12	$\$60$
13	$\$65$
14	$\$70$
15	$\$75$

Company 2	
Day	Penalty
1	$\$0.01$
2	$\$0.02$
3	$\$0.04$
4	$\$0.08$
5	$\$0.16$
6	$\$0.32$
7	$\$0.64$
8	$\$1.28$
9	$\$2.56$
10	$\$5.12$
11	$\$10.24$
12	$\$20.48$
13	$\$40.96$
14	$\$81.92$
15	$\$163.84$

1. **Which company has a greater 15-day late charge?**

   ***Company** 2*

2. **Describe how the amount of the late charge changes from any given day to the next successive day in both Companies 1 and 2.**

   For Company** 1**, the change from any given day to the next successive day is an increase by** $\$5$**. For Company** 2**, the change from any given day to the next successive day is an increase by a factor of** 2**.

3. **How much would the late charge have been after 20 days under Company 2?**

   $\$5,242.88$

Then discuss the following:

- Write a formula for the sequence that models the data in the table for Company 1.
  - $f(n) = 5n$, *where* n *begins with* 1.
- Is the sequence Arithmetic, Geometric, or neither?
  - *Arithmetic*
- Write a formula for the sequence that models the data in the table for Company 2.
  - $f(n) = 0.01(2)^{n-1}$, *where* n *begins with* 1.
- Is the sequence Arithmetic, Geometric, or neither?
  - *Geometric*
- Which of the two companies would you say grows more quickly? Why?
  - *The penalty in Company* 2 *grows more quickly after a certain time because each time you are multiplying by* 2 *instead of just adding* 5.

Example 1 (5 minutes)

Example 1

Folklore suggests that when the creator of the game of chess showed his invention to the country's ruler, the ruler was highly impressed. He was so impressed, he told the inventor to name a prize of his choice. The inventor, being rather clever, said he would take a grain of rice on the first square of the chessboard, two grains of rice on the second square of the chessboard, four on the third square, eight on the fourth square, and so on, doubling the number of grains of rice for each successive square. The ruler was surprised, even a little offended, at such a modest prize, but he ordered his treasurer to count out the rice.

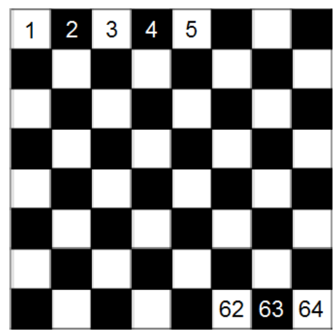
a. Why is the ruler "surprised"? What makes him think the inventor requested a "modest prize"?

The ruler is surprised because he hears a few grains mentioned—it seems very little, but he does not think through the effect of doubling each collection of grains; he does not know that the amount of needed rice will grow exponentially.

The treasurer took more than a week to count the rice in the ruler's store, only to notify the ruler that it would take more rice than was available in the entire kingdom. Shortly thereafter, as the story goes, the inventor became the new king.

b. Imagine the treasurer counting the needed rice for each of the 64 squares. We know that the first square is assigned a single grain of rice, and each successive square is double the number of grains of rice of the former square. The following table lists the first five assignments of grains of rice to squares on the board. How can we represent the grains of rice as exponential expressions?

Square #	Grains of Rice	Exponential Expression
1	1	2^0
2	2	2^1
3	4	2^2
4	8	2^3
5	16	2^4



c. Write the exponential expression that describes how much rice is assigned to each of the last three squares of the board.

Square #	Exponential Expression
62	2^{61}
63	2^{62}
64	2^{63}

Discussion (10 minutes): Exponential Formulas

Ask students to consider how the exponential expressions of Example 1, part (b) relate to one another.

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- Why is the base of the expression 2?
  - *Since each successive square has twice the amount of rice as the former square, the factor by which the rice increases is a factor of* 2.
- What is the explicit formula for the sequence that models the number of rice grains in each square? Use n to represent the number of the square and $f(n)$ to represent the number of rice grains assigned to that square.
  - $f(n) = 2^{(n-1)}$, *where* $f(n)$ *represents the number of rice grains belonging to each square, and* n *represents the number of the square on the board.*
- Would the formula $f(n) = 2^n$ work? Why or why not?
  - *No, the formula is supposed to model the numbering scheme on the chessboard corresponding to the story.*
- What would have to change for the formula $f(n) = 2^n$ to be appropriate?
  - *If the first square started with* 2 *grains of rice and doubled thereafter, or if we numbered the squares as starting with square number* 0 *and ending on square* 63, *then* $f(n) = 2^n$ *would be appropriate.*
- Suppose instead that the first square did not begin with a single grain of rice but with 5 grains of rice, and then the number of grains was doubled with each successive square. Write the sequence of numbers representing the number of grains of rice for the first five squares.
  - $5, 10, 20, 40, 80$
- Suppose we wanted to represent these numbers using exponents? Would we still require the use of the powers of 2?
  - *Yes.* $5 = 5(2^0), 10 = 5(2^1), 20 = 5(2^2), 40 = 5(2^3), 80 = 5(2^4)$
- Generalize the pattern of these exponential expressions into an explicit formula for the sequence. How does it compare to the formula in the case where we began with a single grain of rice in the first square?
  - $f(n) = 5(2^{n-1})$, *the powers of* 2 *cause the doubling effect, and the* 5 *represents the initial* 5 *grains of rice.*
- Generalize the formula even further. Write a formula for a sequence that allows for any possible value for the number of grains of rice on the first square.
  - $f(n) = a2^{n-1}$, *where* a *represents the number of rice grains on the first square.*
- Generalize the formula even further. What if instead of doubling the number of grains, we wanted to triple or quadruple them?
  - $f(n) = ab^{n-1}$, *where* a *represents the number of rice grains on the first square and* b *represents the factor by which the number of rice grains is multiplied on each successive square.*
- Is the sequence for this formula Geometric, Arithmetic, or neither?
  - *Geometric*

Example 2 (10 minutes)

Note that students may, or may not, connect the points in their graphs with a smooth line or curve as shown below. Be clear with the students that we were only asked to graph the points but that it is natural to recognize the form that the points take on by modeling them with the line or curve.

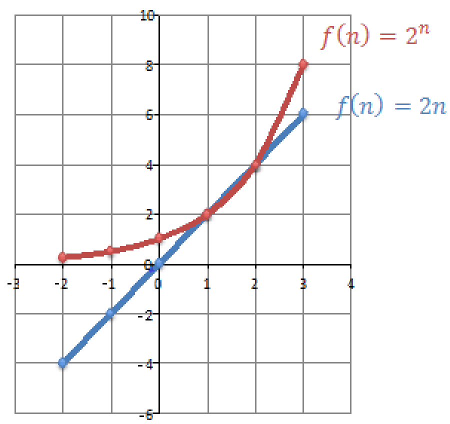
Example 2

Let us understand the difference between $f(n) = 2n$ and $f(n) = 2^n$.

a. **Complete the tables below, and then graph the points $(n, f(n))$ on a coordinate plane for each of the formulas.**

n	$f(n) = 2n$
-2	-4
-1	-2
0	0
1	2
2	4
3	6

n	$f(n) = 2^n$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



b. **Describe the change in each sequence when n increases by 1 unit for each sequence.**

For the sequence $f(n) = 2n$, for every increase in n by 1 unit, the $f(n)$ value increases by 2 units. For the sequence $f(n) = 2^n$, for every increase in n by 1 unit, the $f(n)$ value increases by a factor of 2.

Exercise 1 (7 minutes)

Students should attempt Exercises 1 and 2 independently and then share responses as a class.

Exercise 1

A typical thickness of toilet paper is 0.001 inches. This seems pretty thin, right? Let's see what happens when we start folding toilet paper.

a. How thick is the stack of toilet paper after 1 fold? After 2 folds? After 5 folds?

After 1 fold: $0.001(2^1) = 0.002''$

After 2 folds: $0.001(2^2) = 0.004''$

After 5 folds: $0.001(2^5) = 0.032''$

b. Write an explicit formula for the sequence that models the thickness of the folded toilet paper after n folds.

$f(n) = 0.001(2^n)$

c. After how many folds will the stack of folded toilet paper pass the 1 foot mark?

After 14 folds.

d. The moon is about $240,000$ miles from Earth. Compare the thickness of the toilet paper folded 50 times to the distance from Earth.

Toilet paper folded 50 times is approximately $17,769,885$ miles thick. That is approximately 74 times the distance between the Earth and the moon.

Watch the following video "[How folding paper can get you to the moon](http://www.youtube.com/watch?v=AmFMJC45f1Q)" (http://www.youtube.com/watch?v=AmFMJC45f1Q)

Exercise 2 (3 minutes)

Exercise 2

A rare coin appreciates at a rate of 5.2% a year. If the initial value of the coin is $\$500$, after how many years will its value cross the $\$3,000$ mark? Show the formula that will model the value of the coin after t years.

The value of the coin will cross the $\$3,000$ mark between 35 and 36 years; $f(t) = 500(1.052)^t$.

Closing (2 minutes)

- Consider the sequences $G(n) = ab^n$, where n begins at 0 and $A(n) = a + bn$, where n begins at 0. Assume that $b > 1$.
- Which sequence will have the larger 0^{th} term? Does it depend on what values are chosen for a and b?
  - *Both sequences will have the same 0^{th} term; the term will be a, regardless of what values are chosen for a and b.*

Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 5: The Power of Exponential Growth

Exit Ticket

Chain emails are emails with a message suggesting you will have good luck if you forward the email on to others. Suppose a student started a chain email by sending the message to 3 friends and asking those friends to each send the same email to 3 more friends exactly 1 day after they received it.

a. Write an explicit formula for the sequence that models the number of people who will receive the email on the n^{th} day. (Let the first day be the day the original email was sent.) Assume everyone who receives the email follows the directions.

b. Which day will be the first day that the number of people receiving the email exceeds 100?

Exit Ticket Sample Solutions

Chain emails are emails with a message suggesting you will have good luck if you forward the email on to others. Suppose a student started a chain email by sending the message to 3 friends and asking those friends to each send the same email to 3 more friends exactly 1 day after they received it.

a. Write an explicit formula for the sequence that models the number of people who will receive the email on the n^{th} day. (Let the first day be the day the original email was sent.) Assume everyone who receives the email follows the directions.

$f(n) = 3^n, \quad n \geq 1$

b. Which day will be the first day that the number of people receiving the email exceeds 100?

On the 5^{th} day.

Problem Set Sample Solutions

1. **A bucket is put under a leaking ceiling. The amount of water in the bucket doubles every minute. After 8 minutes, the bucket is full. After how many minutes is the bucket half full?**

   7 minutes

2. **A three-bedroom house in Burbville was purchased for $\$190,000$. If housing prices are expected to increase 1.8% annually in that town, write an explicit formula that models the price of the house in t years. Find the price of the house in 5 years.**

   $f(t) = 190,000(1.018)^t$*, so* $f(5) = 190,000(1.018)^5 = \$207,726.78.$

3. **A local college has increased the number of graduates by a factor of 1.045 over the previous year for every year since 1999. In 1999, 924 students graduated. What explicit formula models this situation? Approximately how many students will graduate in 2014?**

   $f(t) = 924(1.045)^t$*, so* $f(15) = 924(1.045)^{15} = 1,788$ *graduates are expected in 2014.*

4. **The population growth rate of New York City has fluctuated tremendously in the last 200 years, the highest rate estimated at 126.8% in 1900. In 2001, the population of the city was $8,008,288$, up 2.1% from 2000. If we assume that the annual population growth rate stayed at 2.1% from the year 2000 onward, in what year would we expect the population of New York City to have exceeded ten million people? Be sure to include the explicit formula you use to arrive at your answer.**

   $f(t) = 8,008,288(1.021)^t$. *Based on this formula, we can expect the population of New York City to exceeded ten million people in 2012.*

5. **In 2013, a research company found that smartphone shipments (units sold) were up 32.7% worldwide from 2012, with an expectation for the trend to continue. If 959 million units were sold in 2013, how many smartphones can be expected to be sold in 2018 at the same growth rate? (Include the explicit formula for the sequence that models this growth.) Can this trend continue?**

   $f(t) = 959(1.327)^t$; $f(5) = 959(1.327)^5 = 3,946$. *Approximately* 3.95 *billion units are expected to be sold in 2018. No. There are a finite number of people on Earth, so this trend cannot continue.*

6. Two band mates have only 7 days to spread the word about their next performance. Jack thinks they can each pass out 100 fliers a day for 7 days and they will have done a good job in getting the news out. Meg has a different strategy. She tells 10 of her friends about the performance on the first day and asks each of her 10 friends to each tell a friend on the second day and then everyone who has heard about the concert to tell a friend on the third day and so on, for 7 days. Make an assumption that students make sure they are telling someone who has not already been told.

   a. Over the first 7 days, Meg's strategy will reach fewer people than Jack's. Show that this is true.

      Jack's strategy: $J(t) = 100/day \times 7\ days = 700$ *people will know about the concert.*

      Meg's strategy: $M(t) = 10(2)^{t-1}$; $M(7) = 640$ *people will know about the concert.*

   b. If they had been given more than 7 days, would there be a day on which Meg's strategy would begin to inform more people than Jack's strategy? If not, explain why not. If so, which day would this occur on?

      On the 8^{th} *day, Meg's strategy would reach more people than Jack's:* $J(8) = 800$; $M(8) = 1280$.

   c. Knowing that she has only 7 days, how can Meg alter her strategy to reach more people than Jack does?

      She can ask her ten initial friends to tell two people each and let them tell two other people on the next day, etc.

7. On June 1, a fast-growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered, and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.

   a. When will the lake be covered half way?

      June 29

   b. On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?

      On June 26, the lake will only be 6.25% *covered. To the casual observer, it will be hard to imagine such a jump between this small percent of coverage to* 100% *coverage in merely* 4 *more days.*

   c. On June 29, a clean-up crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?

      It only takes care of the problem for a week:

      June 29–1%

      June 30–2%

      July 1–4%

      July 2–8%

      July 3–16%

      July 4–32%

      July 5–64%

      By July 6, the lake will be completely covered with algae.

   d. Write an explicit formula for the sequence that models the percentage of the surface area of the lake that is covered in algae, a, given the time in days, t, that has passed since the algae was introduced into the lake.

      $f(t) = a\left(2^{(t-1)}\right)$

8. **Mrs. Davis is making a poster of math formulas for her students. She takes the 8.5 in. × 11 in. paper she printed the formulas on to the photocopy machine and enlarges the image so that the length and the width are both 150% of the original. She enlarges the image a total of 3 times before she is satisfied with the size of the poster. Write an explicit formula for the sequence that models the area of the poster, A, after n enlargements. What is the area of the final image compared to the area of the original, expressed as a percent increase and rounded to the nearest percent?**

The area of the original piece of paper is 93.5 in². Increasing the length and width by a factor of 1.5 increases the area by a factor of 2.25. Thus, $A(n) = 93.5(2.25)^n$. The area after 3 iterations is approximated by $93.5(11.39)$ for a result of 1065 in². The percent increase was 1039%.