



Lesson 2: Recursive Formulas for Sequences

Student Outcomes

- Students write sequences with recursive and explicit formulas.

Lesson Notes

In this lesson, students will work on recursive formulas building on the ideas that were introduced in Module 1, Lessons 26 and 27 (The Double and Add 5 Game).

Classwork

Opening (2 minutes)

Remind students of their previous experiences with sequences.

- In Lesson 1, we worked on writing explicit formulas for sequences. Explicit formulas relate each term in a sequence directly to its placement in the sequence. This type of formula allows us to *jump* to any term of the sequence by simply replacing n with a specific number and evaluating the expression that describes the n^{th} term of the sequence.
- Today, we will be looking at recursive formulas. You saw these at the end of Module 1 when we played The Double and Add 5 Game.

Example 1 (10 minutes)

Allow students a minute to examine the sequence and to answer part (a). Then, lead the following discussion, building on what was learned in Lesson 1.

Example 1

Consider Akelia's sequence 5, 8, 11, 14, 17,

- a. If you believed in patterns, what might you say is the next number in the sequence?

20 (adding 3 each time)

- She decided to call the sequence the "Akelia" sequence and so chose to use the letter A for naming it.
- When asked to find a formula for this sequence, Akelia wrote the following on a piece of paper: (record this on the board for the students)

$$\begin{aligned} &5 \\ &8 = 5 + 3 \\ &11 = 5 + 3 + 3 = 5 + 2 \times 3 \\ &14 = 5 + 3 + 3 + 3 = 5 + 3 \times 3 \end{aligned}$$

MP.8

- Can you use her reasoning to help you write a formula for Akelia’s sequence?
 - $A(n) = 5 + 3(n - 1)$
- Record the formula in your student materials.

b. Write a formula for Akelia’s sequence.
 $A(n) = 5 + 3(n - 1)$

- What does $A(n)$ represent again?
 - *It means the n^{th} term of the sequence.*

Perhaps replace the $A(n)$ of the formula with the words “the n^{th} term of Akelia’s sequence.” Continue to emphasize the point that $A(n)$ does not mean multiply A and n .

- Can you explain Akelia’s formula and why it works?
 - *To find each term in the sequence, you are adding 3 one less time than the term number. To get the 1st term, you add three zero times. To get the 2nd term, you add 3 one time. To get the 5th term, you add 3 four times. In her formula, she is starting with $n = 1$.*
- Record the explanation in your student materials.

c. Explain how each part of the formula relates to the sequence.
To find each term in the sequence, you are adding 3 one less time than the term number. To get the 1st term, you add three zero times. To get the 2nd term, you add 3 one time. To get the 5th term, you add 3 four times.

- Akelia’s formula is an explicit formula. You can use the formula to find the value of any term you want without having to know the value of the term before it. For example, if you wanted to know the 88th term, just substitute 88 for n and evaluate.
- When Johnny saw the sequence, he wrote the following: $A(n + 1) = A(n) + 3$ for $n \geq 1$ and $A(1) = 5$. (Display the formula on the board.)
- But what does the $A(n + 1)$ mean? Look back at the sequence. (Write the following on the board.)

$$5$$

$$8 = 5 + 3$$

$$11 = 8 + 3$$

$$14 = 11 + 3$$

$$17 = 14 + 3$$

- What do we call the 5th term?
 - $A(5)$
- How could we find the 5th term in terms of the 4th term?
 - $A(4) + 3$
- If we want the 6th term in terms of the 5th term?
 - $A(5) + 3$
- If we want the $(n + 1)^{\text{th}}$ term in terms of the n^{th} term?
 - $A(n) + 3$

MP.8

- Now explain what Johnny’s formula means.
 - *His formula is saying to find any term in the sequence just add 3 to the previous term. For example, to find the 12th term, add 3 to the 11th term: $A(12) = A(11) + 3$. To find the 50th term, add 3 to the 49th term: $A(50) = A(49) + 3$. To find the $(n + 1)$ th term, add 3 to the n th term.*
- Record the explanation in your student materials.

d. Explain Johnny’s formula.

His formula is saying to find any term in the sequence just add 3 to the term before it. For example, to find the 12th term, add 3 to the 11th term: $A(12) = A(11) + 3$. To find the 50th term, add 3 to the 49th term. To find the $(n + 1)$ th term, add 3 to the n th term. It is critical that the value of the very first term be specified; we need it to get started to find the values of all the other terms.

- The statement $A(n + 1) = A(n) + 3$ is a recursive formula. A recursive formula relates a term in the sequence to the preceding term or terms of the sequence.

(Note: For students that struggle to understand notation involving n , $n + 1$, $n - 1$, consider quick exercises of this type: “If we start with 3, what expression would name the next whole number, 4?” $(3 + 1)$. “What expression would name the previous whole number, 2?” $(3 - 1)$.)

- Would it be equivalent to write the sequence as $A(n) = A(n - 1) + 3$? Why or why not?
 - *Yes, $A(n - 1)$ is the term before $A(n)$ just like $A(n)$ is the term before $A(n + 1)$. Both formulas are saying that to find any term in the sequence, add three to the previous term.*

Warn students that they will see recursive formulas written in both of these ways.

Again, caution students that $A(n + 1)$ is not $A \cdot (n + 1)$.

- Why does Akelia’s formula have a “times 3” in it, while Johnny’s formula has a “plus 3”?
 - *Akelia’s formula specifies the n th term directly as an expression in n . Johnny’s formula evaluates the $(n + 1)$ th term by using the n th term, which means he only has to observe the rule that takes one term to the next consecutive term. In this case, the rule is to add 3 to the previous term.*
- If we wanted the 200th term of the sequence, which formula would be more useful?
 - *Akelia’s—just fill in 200 for n .*
- If we wanted to know how the sequence changes from one term to the next, which formula would be more useful?
 - *Johnny’s recursive formula would be more useful.*
- Using Johnny’s recursive formula, what would we need to know if we wanted to find the 200th term?
 - *We would need to know the 199th term.*

Exercises 1–2 (8 minutes)

As students work through Exercises 1 and 2, circulate the room making sure that students understand the notation. Ask students to read the notation aloud and explain the meaning. Debrief by having students share answers.

- Throughout these exercises, ask students to translate the sequences into words:
 - *$A(n + 1) = A(n) - 3$ is a sequence where each term is three less than the term before it.*

MP.6

Exercises 1–2

1. Akelia, in a playful mood, asked Johnny: What would happen if we change the “+” sign in your formula to a “–” sign? To a “ \times ” sign? To a “ \div ” sign?
- a. What sequence does $A(n + 1) = A(n) - 3$ for $n \geq 1$ and $A(1) = 5$ generate?
5, 2, –1, –4, ...
- b. What sequence does $A(n + 1) = A(n) \cdot 3$ for $n \geq 1$ and $A(1) = 5$ generate?
5, 15, 45, 135, ...
- c. What sequence does $A(n + 1) = A(n) \div 3$ for $n \geq 1$ and $A(1) = 5$ generate?
 $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, \dots$
2. Ben made up a recursive formula and used it to generate a sequence. He used $B(n)$ to stand for the n^{th} term of his recursive sequence.
- a. What does $B(3)$ mean?
It is the third term of Ben’s sequence.
- b. What does $B(m)$ mean?
It is the m^{th} term of Ben’s sequence.
- c. If $B(n + 1) = 33$ and $B(n) = 28$, write a possible recursive formula involving $B(n + 1)$ and $B(n)$ that would generate 28 and 33 in the sequence.
 $B(n) = B(n - 1) + 5$. (Note that this is not the only possible answer; it assumes the sequence is arithmetic and is probably the most obvious response students will give. If the sequence were geometric, the answer could be written $B(n + 1) = \left(\frac{33}{28}\right)B(n)$.)
- d. What does $2B(7) + 6$ mean?
It is 2 times the 7th term of Ben’s sequence plus 6.
- e. What does $B(n) + B(m)$ mean?
It is the sum of n^{th} term of Ben’s sequence plus the m^{th} term of Ben’s sequence.
- f. Would it necessarily be the same as $B(n + m)$?
No, adding two terms of a sequence is not the same as adding two of the term numbers and then finding that term of a sequence. Consider, for example, the sequence 1, 3, 5, 7, 9, 11, 13, Adding the 2nd and 3rd terms does not give you the 5th term.
- g. What does $B(17) - B(16)$ mean?
It is the 17th term of Ben’s sequence minus the 16th term of Ben’s sequence.

- When writing a recursive formula, what piece of information is necessary to include along with the formula?
 - *The value of the initial term with which the sequence starts, which is usually identified as the first term and indexed by the term number 1.*

Point out to students that there is no hard-and-fast requirement that all recursive sequences start with index at 1. In some cases, it is convenient to start the index at 0 (as was done in the Double and Add 5 Game). However, in this sequence of lessons, we are mostly concerned with building up to the idea of function, so we will mostly stay with sequences starting at index 1.

- What additional piece of information is needed when writing a recursive formula?
 - *We need to describe what n the formula holds for. For example, Johnny's formula $A(n + 1) = A(n) + 3$ does not hold for $n = -5$.*

Example 2 (5 minutes)

Point out the new notation of using a subscript rather than parentheses. Assure students that the two notations are essentially the same and that they will see both throughout the unit. Give students a few minutes to complete the problem.

Example 2

Consider a sequence given by the formula $a_n = a_{n-1} - 5$, where $a_1 = 12$ and $n \geq 1$.

- a. List the first five terms of the sequence.

$12, 7, 2, -3, -8$

- b. Write an explicit formula.

$a_n = 12 - 5(n - 1)$ for $n \geq 1$.

- c. Find a_6 and a_{100} of the sequence.

$a_6 = -13$ $a_{100} = -483$

- What type of formula is given in the question: recursive or explicit?
 - *Recursive because it relates a term in the sequence to the term before it.*
- Which formula did you use to find a_6 ? a_{100} ?
 - *Probably recursive to find the 6th term. Since the 5th term was known, it makes sense to just continue the sequence to find the 6th term. The explicit formula is the easiest to use to find the 100th term. In order to use the recursive formula, we would need to know the 99th term.*

Exercises 3–6 (12 minutes)

Give students time to work through the exercises either individually or in pairs, circulating the room to make sure students are recognizing the differences between the two types of formulas and are using correct notation.

Scaffolding:

- Early finishers could be asked to research various places in nature where the Fibonacci sequence can be seen and then briefly share with the class.
- Students with computer experience may be familiar with spreadsheets. Show them how recursive formulas can easily be input in a spreadsheet.

Exercises 3–6

3. One of the most famous sequences is the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$f(n + 1) = f(n) + f(n - 1), \text{ where } f(1) = 1, f(2) = 1, \text{ and } n \geq 2.$$

How is each term of the sequence generated?

By adding the two preceding terms.

4. For each sequence below, an explicit formula is given. Write the first 5 terms of each sequence. Then, write a recursive formula for the sequence.

a. $a_n = 2n + 10$ for $n \geq 1$

12, 14, 16, 18, 20

$$a_{n+1} = a_n + 2, \text{ where } a_1 = 12 \text{ and } n \geq 1$$

b. $a_n = \left(\frac{1}{2}\right)^{n-1}$ for $n \geq 1$

1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$

$$a_{n+1} = a_n \div 2, \text{ where } a_1 = 1 \text{ and } n \geq 1$$

5. For each sequence, write *either* an explicit or recursive formula.

a. 1, -1, 1, -1, 1, -1, ...

$$a_{n+1} = -a_n, \text{ where } a_1 = 1 \text{ and } n \geq 1 \text{ or } f(n) = (-1)^{n+1}, \text{ where } n \geq 1$$

b. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

$$f(n) = \frac{n}{n+1} \text{ and } n \geq 1$$

6. Lou opens a bank account. The deal he makes with his mother is that if he doubles the amount that was in the account at the beginning of each month by the end of the month, she will add an additional \$5 to the account at the end of the month.

a. Let $A(n)$ represent the amount in the account at the beginning of the n^{th} month. Assume that he does, in fact, double the amount every month. Write a recursive formula for the amount of money in his account at the beginning of the $(n + 1)^{\text{th}}$ month.

$$A(n + 1) = 2A(n) + 5, \text{ where } n \geq 1 \text{ and } A(1) \text{ is the initial amount.}$$

b. What is the least amount he could start with in order to have \$300 by the beginning of the 3rd month?

$$\begin{aligned} A(3) &= 2 \cdot A(2) + 5 \\ A(3) &= 2 \cdot [2 \cdot A(1) + 5] + 5 \\ 300 &\leq 2 \cdot [2 \cdot A(1) + 5] + 5 \\ 300 &\leq 4 \cdot A(1) + 15 \\ \frac{285}{4} &\leq A(1) \\ \$71.25 &\leq A(1) \end{aligned}$$

- Notice that in the Fibonacci sequence, each term depends on the two previous terms. This means we had to know the first two terms in order to start the sequence. Point out that an explicit formula would be much more complicated to come up with in this case.
- For Exercises 5(a) and 5(b), which type of formula did you write?
 - For Exercise 5(a), either formula was fairly easy to come up with. For Exercise 5(b), an explicit formula is easier to write. The recursive formula would be pretty tough to come up with. If you want to share the recursive formula for Exercise 5(b) just for fun, it is $f(n + 1) = \frac{(n+1)^2 f(n)}{n(n+2)}$.
- Does Exercise 6 seem familiar?
 - We are revisiting the Double and Add 5 Game from Module 1!

Closing (3 minutes)

- What are two types of formulas that can be used to represent a sequence?
 - *Explicit and recursive.*

Go over the definition of each as given in the Lesson Summary. If time permits, have students put an example next to each definition, and then share a few with the class.

- What information besides the formula would you need in order to write each of these two types of formulas?
 - *To write an explicit formula, you need to know what integer you are using for the first term number.*
 - *To write a recursive formula, you need to know what the first term is, or first several terms are, depending on the recursive relation.*

Lesson Summary

Recursive Sequence: An example of a *recursive sequence* is a sequence that (1) is defined by specifying the values of one or more initial terms and (2) has the property that the remaining terms satisfy a recursive formula that describes the value of a term based upon an expression in numbers, previous terms, or the index of the term.

An explicit formula specifies the n^{th} term of a sequence as an expression in n .

A recursive formula specifies the n^{th} term of a sequence as an expression in the previous term (or previous couple of terms).

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 2: Recursive Formulas for Sequences

Exit Ticket

1. Consider the sequence following a “minus 8” pattern: 9, 1, -7 , -15 ,
 - a. Write an explicit formula for the sequence.

 - b. Write a recursive formula for the sequence.

 - c. Find the 38th term of the sequence.

2. Consider the sequence given by the formula $a(n + 1) = 5a(n)$ and $a(1) = 2$ for $n \geq 1$.
 - a. Explain what the formula means.

 - b. List the first 5 terms of the sequence.

Exit Ticket Sample Solutions

- Consider the sequence following a “minus 8” pattern: 9, 1, -7, -15,
 - Write an explicit formula for the sequence.
 $f(n) = 9 - 8(n - 1)$ for $n \geq 1$
 - Write a recursive formula for the sequence.
 $f(n + 1) = f(n) - 8$ and $f(1) = 9$ for $n \geq 1$
 - Find the 38th term of the sequence.
 $f(38) = 9 - 8(37) = -287$
- Consider the sequence given by the formula $a(n + 1) = 5a(n)$ and $a(1) = 2$ for $n \geq 1$.
 - Explain what the formula means.
The first term of the sequence is 2. Each subsequent term of the sequence is found by multiplying the previous term by 5.
 - List the first 5 terms of the sequence.
2, 10, 50, 250, 1250

Problem Set Sample Solutions

For problems 1-4, list the first five terms of each sequence.

- | | |
|---|---|
| 1. $a_{n+1} = a_n + 6$, where $a_1 = 11$ for $n \geq 1$
11, 17, 23, 29, 35 | 2. $a_n = a_{n-1} \div 2$, where $a_1 = 50$ for $n \geq 2$
50, 25, 12.5, 6.25, 3.125 |
| 3. $f(n + 1) = -2f(n) + 8$ and $f(1) = 1$ for $n \geq 1$
1, 6, -4, 16, -24 | 4. $f(n) = f(n - 1) + n$ and $f(1) = 4$ for $n \geq 2$
4, 6, 9, 13, 18 |

For Problems 5-10, write a recursive formula for each sequence given or described below.

- | | |
|--|---|
| 5. It follows a “plus one” pattern: 8, 9, 10, 11, 12,
$f(n + 1) = f(n) + 1$, where $f(1) = 8$ and $n \geq 1$ | 6. It follows a “times 10” pattern: 4, 40, 400, 4000,
$f(n + 1) = 10f(n)$, where $f(1) = 4$ and $n \geq 1$ |
| 7. It has an explicit formula of $f(n) = -3n + 2$ for $n \geq 1$.
$f(n + 1) = f(n) - 3$, where $f(1) = -1$ and $n \geq 1$ | 8. It has an explicit formula of $f(n) = -1(12)^{n-1}$ for $n \geq 1$.
$f(n + 1) = 12f(n)$, where $f(1) = -1$ for $n \geq 1$ |
9. Doug accepts a job where his starting salary will be \$30,000 per year, and each year he will receive a raise of \$3,000.
 $D_{n+1} = D_n + 3000$, where $D_1 = 30,000$ and $n \geq 1$
10. A bacteria culture has an initial population of 10 bacteria, and each hour the population triples in size.
 $B_{n+1} = 3B_n$, where $B_1 = 10$ and $n \geq 1$