



Lesson 15: Piecewise Functions

Student Outcomes

- Students examine the features of piecewise functions including the absolute value function and step functions.
- Students understand that the graph of a function f is the graph of the equation $y = f(x)$.

Lesson Notes

This lesson has two main purposes: The first is to continue the work from Lessons 11–13 regarding the interplay between graphs, equations, and functions; the second is to expose students to piecewise functions in general and the absolute value and step functions, specifically. Lessons 12 and 13 established the meaning of the graph of a function and the graph of the equation $y = f(x)$. This lesson continues to clarify that these two sets are one and the same. Students consider two important functions used in later lessons and classes: the absolute value function and the greatest integer function.

Classwork

Opening (2 minutes)

Recall that the absolute value of a number is the distance from 0 of a point on the number line. Because we are measuring distance, the absolute value of a nonzero number is always positive. For example, $|-3| = 3$ because the point -3 , located 3 units to the left of 0 on the real number line, is 3 units away from 0. Absolute value can also be used to define the distance between any two points on the real number line. For example, $|5 - 8| = 3$ because there are 3 units between the numbers 5 and 8 on the real number line.

Opening Exercise (3 minutes) (optional)

Opening Exercise

For each real number a , the *absolute value of a* is the distance between 0 and a on the number line and is denoted $|a|$.

1. Solve each one variable equation.

a. $|x| = 6$
 $\{-6, 6\}$

b. $|x - 5| = 4$
 $\{9, 1\}$

c. $2|x + 3| = -10$
No solution.

2. Determine at least five solutions for each two-variable equation. Make sure some of the solutions include negative values for either x or y .

a. $y = |x|$ $\{(-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2)\}$

b. $y = |x - 5|$ $\{(-1, 6), (0, 5), (1, 4), (5, 0), (6, 1)\}$

c. $x = |y|$ $\{(1, 1), (1, -1), (0, 0), (2, 2), (2, -2)\}$

Scaffolding:

Much of this exploration relies on students accessing their knowledge from the beginning of this unit and from Module 1. Provide additional support as needed to reteach these ideas if students are struggling to work the exploration independently.

Exploratory Challenge 1 (15 minutes)

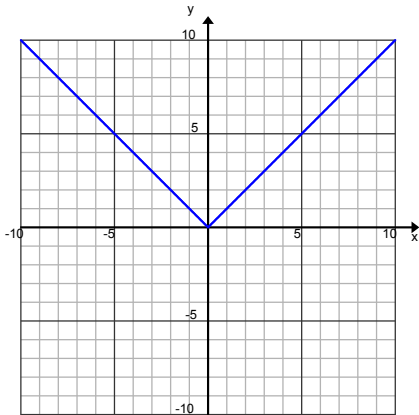
Have students work parts (a)–(d) in small groups. As you circulate, check to see that the groups are creating graphs. Remind them that the domain of the variables for these equations is all real numbers so their graphs should be continuous. Make sure groups are plotting (0,0) for parts (a) and (c) and (5,0) for part (b). After a few minutes, have different groups share their responses. Provide time for groups to revise their graphs as needed.

Part (d) offers an example of MP.6 as students must communicate their findings using precise language. A student example with particularly strong language may be highlighted for the benefit of the class.

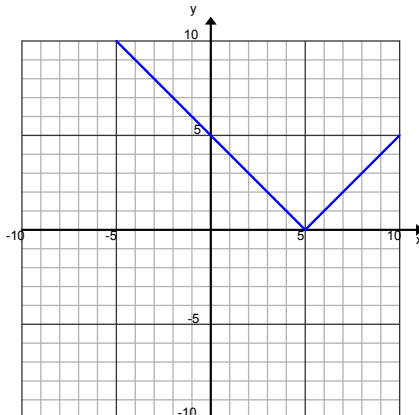
Exploratory Challenge 1

For parts (a)–(c) create graphs of the solution set of each two-variable equation from Opening Exercise 2.

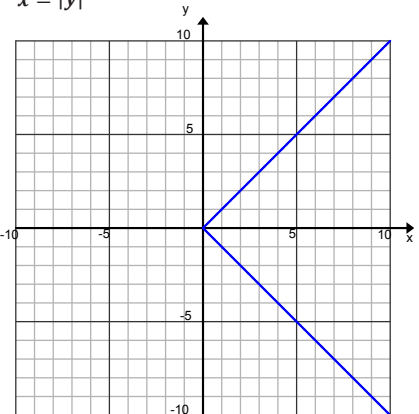
a. $y = |x|$



b. $y = |x - 5|$



c. $x = |y|$



d. Write a brief summary comparing and contrasting the three solution sets and their graphs.

The graphs of parts (a) and (b) are the same except that part (b) has point of the 'vee' (the vertex of angle) at (5, 0) instead of (0, 0). The graph for part (c) looks like a 90° clockwise rotation of the graph from part (a) about the point (0, 0). The points in the solution sets to parts (a) and (b) are a function, but the points in the solution set for part (c) are not.

The next portion asks students to consider their work so far in Module 3. Part (h) makes the connection that the graph of the equation $y = |x|$ and the graph of f , where $f(x) = |x|$ are identical. Part (j) helps students understand that the graph of $y = f(x)$ and the graph of a two-variable equation (in x and y) are only identical if the elements of the equation's solution set define a function in x to begin with.

For parts (e) – (j) consider the function $f(x) = |x|$ where x can be any real number.

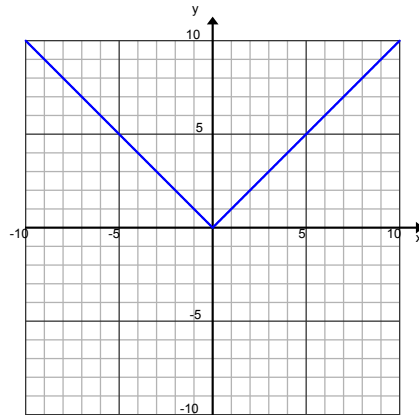
- e. Explain the meaning of the function f in your own words.

This function assigns every real number to its absolute value, which is the distance the point is located from 0 on the real number line. Each number and its opposite will have the same range element. The number 0 will be assigned to 0.

- f. State the domain and range of this function.

Domain: all real numbers. Range: all non-negative real numbers.

- g. Create a graph of the function f . You might start by listing several ordered pairs that represent the corresponding domain and range elements.



- h. How does the graph of the absolute value function compare to the graph of $y = |x|$?

The two graphs are identical. They are identical because each ordered pair in the function would make the equation $y = |x|$ a true number sentence if the domain value were substituted for x and the range value was substituted for y . Therefore the graph of the function is the graph of the solution set of the equation.

- i. Define a function whose graph would be identical to the graph of $y = |x - 5|$?

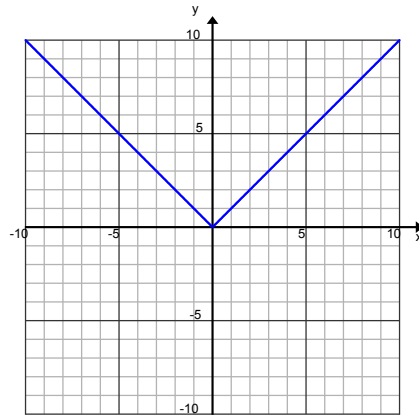
Let $g(x) = |x - 5|$ where x can be any real number.

- j. Could you define a function whose graph would be identical to the graph of $x = |y|$? Explain your reasoning.

No. The graph of $x = |y|$ does not meet the definition of a graph of a function. If it were the graph of a function (say, the function h), it would be the set of ordered pairs $\{(x, h(x)) \mid x \in D\}$, which means there would be only one y -value for each x in the domain D . However, in the graph of $x = |y|$ there is a number x (in fact, there are infinitely many x 's) associated with two different y -values: $(3, 3)$ and $(3, -3)$ are both solutions to the equation $x = |y|$.

As you debrief questions (h)–(j) as a whole group, lead a discussion that includes a summary of the following information. When we create the graph of the solution set to a two-variable equation, we use essentially the same process as when we create the graph of $y = f(x)$. We sift through all the (x, y) pairs in the Cartesian plane and plot only those pairs that make a true number sentence. The difference between the two processes is that when we graph $y = f(x)$, each x value in the domain of f will be paired with only one y value. When graphing a two-variable equation, there is no such restriction placed on the ordered pairs that return a true number sentence. The process of creating the graph of a function f yields the same results as graphing the solution set to the equation $y = f(x)$ except we run through the set of domain values, determine the corresponding range value, and then plot that ordered pair. Since each x in the domain is paired with exactly one y in the range, the resulting graphs will be the same. For this reason, we often use the variable symbol y and the function name $f(x)$ interchangeably when we talk about the graph of a function or two-variable equation solved for y . The caveat is that the two-variable equation must have a solution set where each x is paired with only one y .

- k. Let $f_1(x) = -x$ for $x < 0$ and let $f_2(x) = x$ for $x \geq 0$. Graph the functions f_1 and f_2 on the same Cartesian plane. How does the graph of these two functions compare to the graph in part (g)?



The graph of these two functions when graphed on the same Cartesian plane is identical to the graph of the absolute value function.

Close this portion of the lesson with the following definition of the absolute value function as a piecewise function.

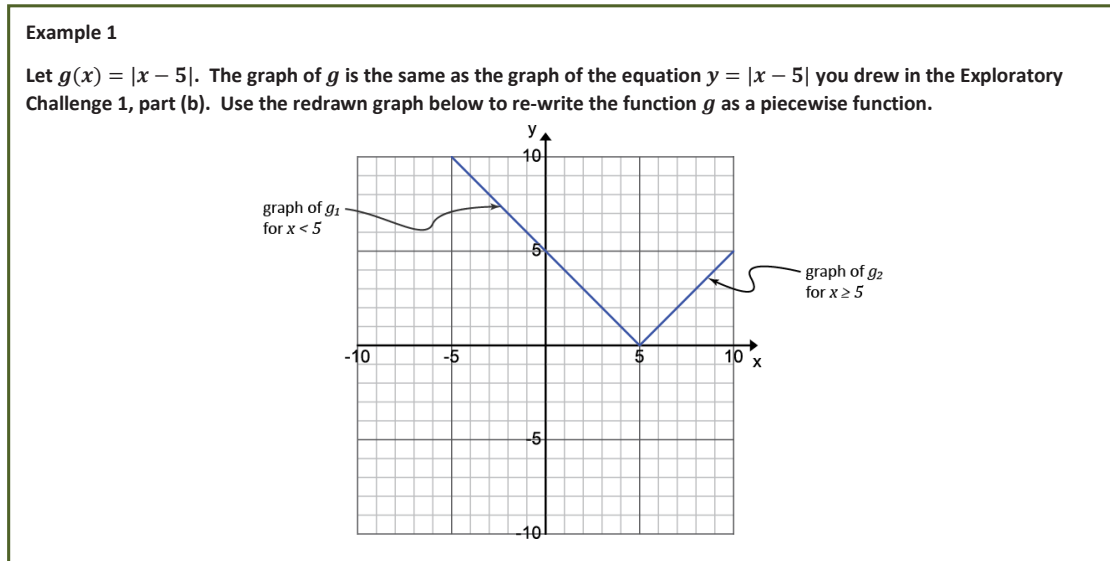
Definition:

The **absolute value function** f is defined by setting $f(x) = |x|$ for all real numbers. Another way to write f is as a **piecewise linear function**:

$$f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

Example 1 (5 minutes)

This example shows students how to express a translation of the absolute value function as a piecewise function. Students create a graph of this function:



Explain that we will need to derive the equations of both lines to write g as a piecewise function.

Label the graph of the linear function with negative slope by g_1 and the graph of the linear function with positive slope by g_2 as in the picture above.

Function g_1 : Slope of g_1 is -1 (why?), and the y -intercept is 5 ; therefore, $g_1(x) = -x + 5$.

Function g_2 : Slope of g_2 is 1 (why?), and the y -intercept is -5 (why?); therefore, $g_2(x) = x - 5$.

Writing g as a piecewise function is just a matter of collecting all of the different “pieces” and the intervals upon which they are defined:

$$g(x) = \begin{cases} -x + 5 & x < 5 \\ x - 5 & x \geq 5 \end{cases}$$

- How does this graph compare to the graph of the translated absolute value function?
 - *The graphs are congruent, but the graph of g has been translated to the right 5 units. (Using terms like “congruent” and “translated” reinforces concepts from 8th grade and prepares students for geometry.)*
- How can you use your knowledge of the graph of $f(x) = |x|$ to quickly determine the graph of $g(x) = |x - 5|$?
 - *Watch where the vertex of the graph of f has been translated. In this case, $g(x) = |x - 5|$ has translated the vertex point from $(0,0)$ to $(5,0)$. Then, graph a line with a slope of -1 for the piece where $x < 5$ and a line with a slope of 1 for the piece where $x > 5$.*
- Can we interpret in words what this function does?
 - *The range values are found by finding the distance between each domain element and the number 5 on the number line.*

Exploratory Challenge 2 (8 minutes)

This exploration introduces the two types of step functions and a third function that is related to them: the floor function (also known as the greatest integer function), the ceiling function, and the sawtooth function. The notation that one often sees for the greatest integer function is $f(x) = \llbracket x \rrbracket$. Gauss first introduced the greatest integer function in the early 1800s. Later, Iverson defined the floor and ceiling functions and introduced the notation you see below in 1962. Both notations are used in mathematics. These functions are used in computer programming languages among other applications. Be sure to explain the notation.

Parts (b) and (c) help students understand how the range values for each function are generated. In part (c), students will begin to understand that all real numbers in the interval have the same y -value. Clarify for students why the interval is closed at the left endpoint and open at the right endpoint. If students are struggling to create graphs, you may need to finish this exploration as a whole class. Before closing the lesson, make sure each student has a correct graph of the functions.

Exploratory Challenge 2

The *floor* of a real number x , denoted by $\lfloor x \rfloor$, is the largest integer not greater than x . The *ceiling* of a real number x , denoted by $\lceil x \rceil$, is the smallest integer not less than x . The *sawtooth* number of a positive number is the “fractional part” of the number that is to the right of its floor on the number line. In general, for a real number x , the sawtooth number of x is the value of the expression $x - \lfloor x \rfloor$. Each of these expressions can be thought of as functions with domain the set of real numbers.

- a. Complete the following table to help you understand how these functions assign elements of the domain to elements of the range. The first and second rows have been done for you.

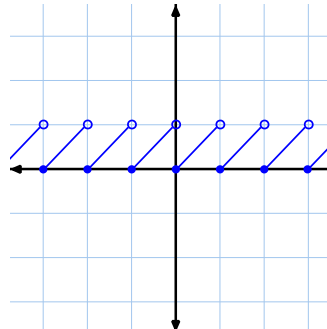
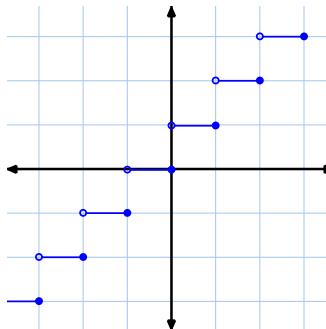
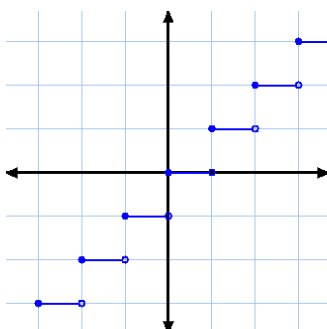
x	$\text{floor}(x) = \lfloor x \rfloor$	$\text{ceiling}(x) = \lceil x \rceil$	$\text{sawtooth}(x) = x - \lfloor x \rfloor$
4.8	4	5	0.8
-1.3	-2	-1	0.7
2.2	2	3	0.2
6	6	6	0
-3	-3	-3	0
$-\frac{2}{3}$	-1	0	$\frac{1}{3}$
π	3	4	$\pi - 3$

- b. Create a graph of each function.

$\text{floor}(x) = \lfloor x \rfloor$

$\text{ceiling}(x) = \lceil x \rceil$

$\text{sawtooth}(x) = x - \lfloor x \rfloor$



- c. For the floor, ceiling, and sawtooth functions, what would be the range values for all real numbers x on the interval $[0, 1)$? The interval $(1, 2]$? The interval $[-2, -1)$? The interval $[1.5, 2.5]$?

Floor: $\{0\}$, *Ceiling:* $\{0, 1\}$, *Sawtooth:* $[0, 1)$.

Floor: $\{1, 2\}$, *Ceiling:* $\{2\}$, *Sawtooth:* $[0, 1)$.

Floor: $\{-2\}$, *Ceiling:* $\{-2, -1\}$, *Sawtooth:* $[0, 1)$.

Floor: $\{1, 2\}$, *Ceiling:* $\{2, 3\}$, *Sawtooth:* $[0, 1)$.

Closing (2 minutes)

- You can use different expressions to define a function over different subsets of the domain. These are called piecewise functions. The absolute value function and step functions can be represented as piecewise functions.
- The graph of a function f and the graph of the equation $y = f(x)$ are the same.

Relevant Vocabulary

Piecewise-Linear Function: Given a number of non-overlapping intervals on the real number line, a (*real*) *piecewise-linear function* is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.

Absolute Value Function: The absolute value of a number x , denoted by $|x|$, is the distance between 0 and x on the number line. The *absolute value function* is the piecewise-linear function such that for each real number x , the value of the function is $|x|$.

We often name the absolute value function by saying, "Let $f(x) = |x|$ for all real numbers x ."

Floor Function: The *floor* of a real number x , denoted by $\lfloor x \rfloor$, is the largest integer not greater than x . The *floor function* is the piecewise-linear function such that for each real number x , the value of the function is $\lfloor x \rfloor$.

We often name the floor function by saying, "Let $f(x) = \lfloor x \rfloor$ for all real numbers x ."

Ceiling Function: The *ceiling* of a real number x , denoted by $\lceil x \rceil$, is the smallest integer not less than x . The *ceiling function* is the piecewise-linear function such that for each real number x , the value of the function is $\lceil x \rceil$.

We often name the ceiling function by saying, "Let $f(x) = \lceil x \rceil$ for all real numbers x ."

Sawtooth Function: The *sawtooth function* is the piecewise-linear function such that for each real number x , the value of the function is given by the expression $x - \lfloor x \rfloor$.

The sawtooth function assigns to each positive number the part of the number (the non-integer part) that is to the right of the floor of the number on the number line. That is, if we let $f(x) = x - \lfloor x \rfloor$ for all real numbers x then

$$f\left(\frac{1}{3}\right) = \frac{1}{3}, f\left(1\frac{1}{3}\right) = \frac{1}{3}, f(1000.02) = 0.02, f(-0.3) = 0.7, \text{ etc.}$$

Exit Ticket (5 minutes)

Name _____

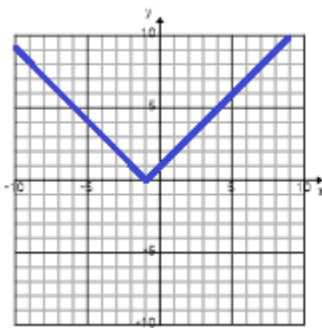
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Lesson 15: Piecewise Functions

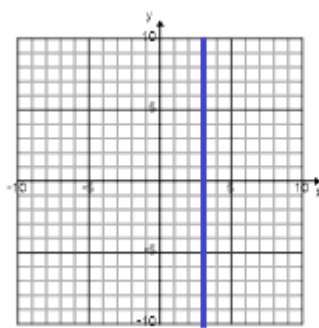
Exit Ticket

Each graph shown below represents the solution set to a two-variable equation.

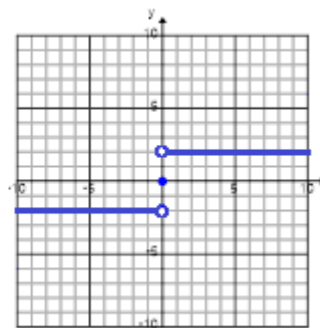
Graph A



Graph B



Graph C



- Which of these graphs could be represented by a function? Explain your reasoning.

- For each one that can be represented by a function, define a piecewise function whose graph would be identical to the solution set shown.

Exit Ticket Sample Solutions

- Which of these graphs could be represented by a function? Explain your reasoning.

Graphs A and C could be represented by a function because each x in the domain is paired with exactly one y in the range.

- For each one that can be represented by a function, define a piecewise function whose graph would be identical to the solution set shown.

Graph A: $f(x) = \begin{cases} -x - 1, & x < -1 \\ x + 1, & x \geq -1 \end{cases}$ Graph C: $f(x) = \begin{cases} -2, & x < 0 \\ 0, & x = 0 \\ 2, & x > 0 \end{cases}$

Problem Set Sample Solutions

These problems build student familiarity with piecewise functions and continue to reinforce the definition of function. The following solutions indicate an understanding of the objectives of this lesson:

- Explain why the sawtooth function, $sawtooth(x) = x - [x]$ for all real numbers x , takes only the “fractional part” of a number when the number is positive.

If you subtract the integer part of a number from the number, only the “fractional part” will remain.

- Let $g(x) = \lceil x \rceil - [x]$ where x can be any real number. In otherwords, g is the difference between the ceiling and floor functions. Express g as a piecewise function.

$$g(x) = \begin{cases} 0 & x \text{ is an integer} \\ 1 & x \text{ is not an integer} \end{cases}$$

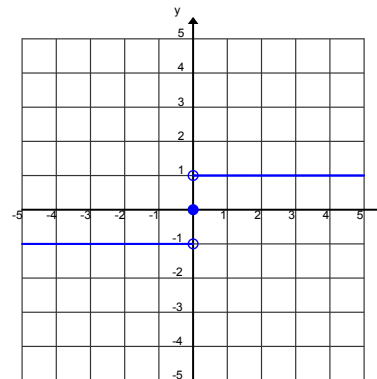
- The Heaviside function is defined using the formula below.

$$H(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Graph this function and state its domain and range.

Domain: All real numbers.

Range: $\{-1, 0, 1\}$.



4. The following piecewise function is an example of a step function.

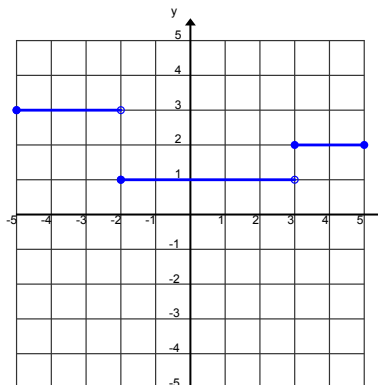
$$S(x) = \begin{cases} 3 & -5 \leq x < -2 \\ 1 & -2 \leq x < 3 \\ 2 & 3 \leq x \leq 5 \end{cases}$$

a. Graph this function and state the domain and range.

Domain: $[-5, 5]$, *Range:* $\{1, 2, 3\}$.

b. Why is this type of function called a step function?

The horizontal line segments step up and down like steps.



5. Let $f(x) = \frac{|x|}{x}$ where x can be any real number except 0.

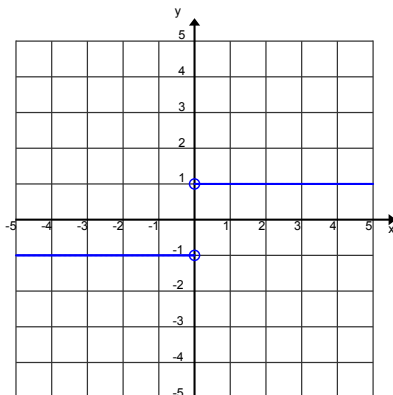
a. Why is the number 0 excluded from the domain of f ?

If $x = 0$ then the expression would not be defined.

b. What is the range of f ?

$\{-1, 1\}$.

c. Create a graph of f .



d. Express f as a piecewise function.

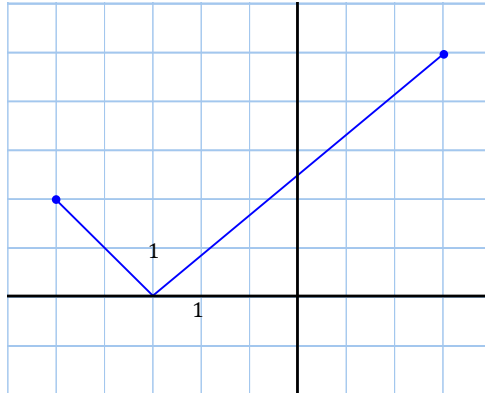
$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

e. What is the difference between this function and the Heaviside function?

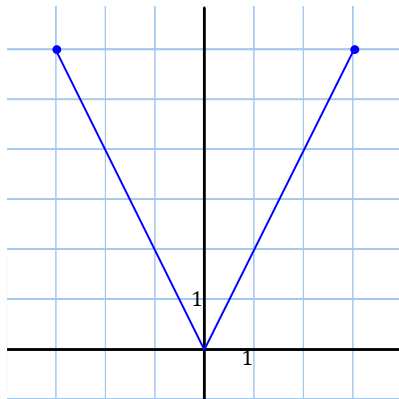
The domain of the Heaviside function is all real numbers. The Heaviside function has a value of 0 when $x = 0$. This function excludes the real number 0 from the domain.

6. Graph the following piecewise functions for the specified domain.

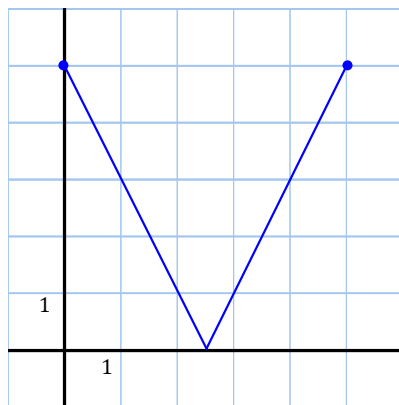
a. $f(x) = |x + 3|$ for $-5 \leq x \leq 3$



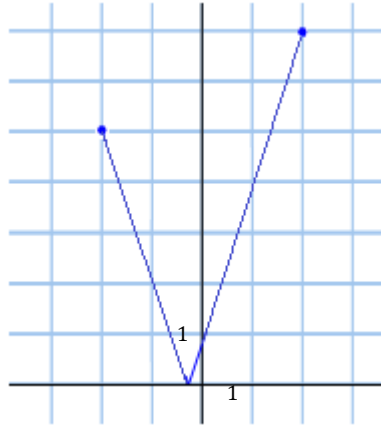
b. $f(x) = |2x|$ for $-3 \leq x \leq 3$



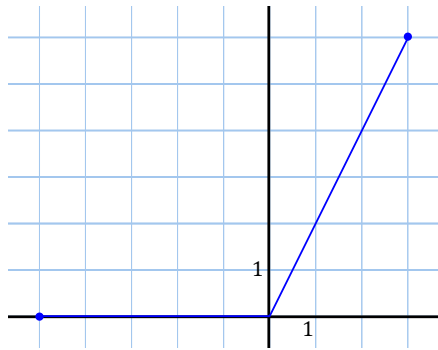
c. $f(x) = |2x - 5|$ for $0 \leq x \leq 5$



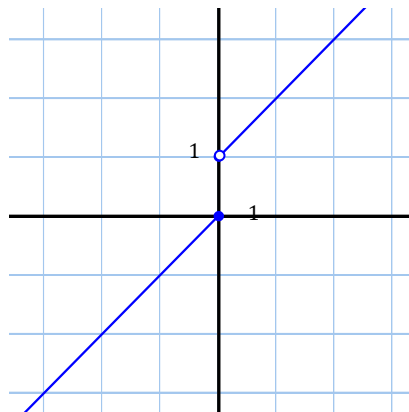
d. $f(x) = |3x + 1|$ for $-2 \leq x \leq 2$



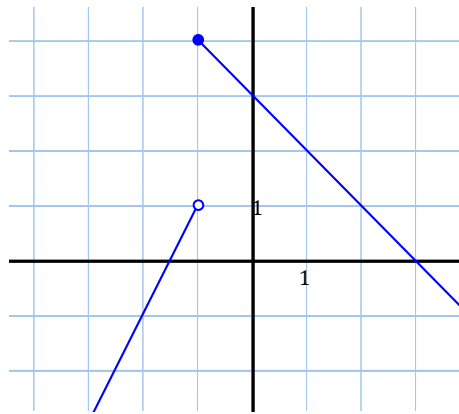
e. $f(x) = |x| + x$ for $-5 \leq x \leq 3$



f. $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

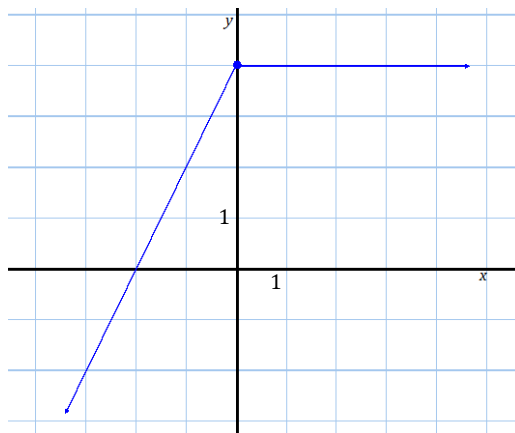


g. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \geq -1 \end{cases}$



7. Write a piecewise function for each graph below.

a.

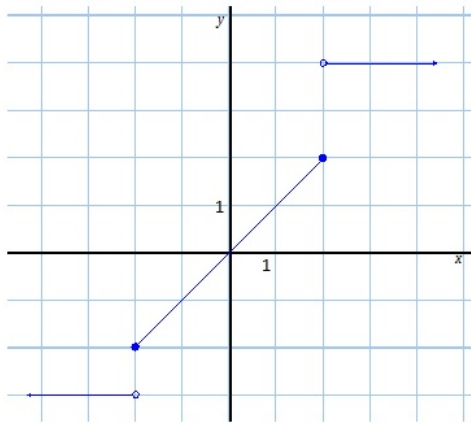


Graph of b

Answer

$$b(x) = \begin{cases} 2x + 4 & x \leq 0 \\ 4 & x \geq 0 \end{cases}$$

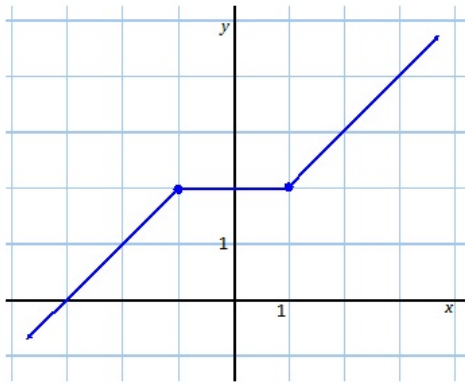
b.



Graph of p

$$p(x) = \begin{cases} -3 & x < -2 \\ 1 & -2 \leq x \leq 2 \\ 4 & x > 2 \end{cases}$$

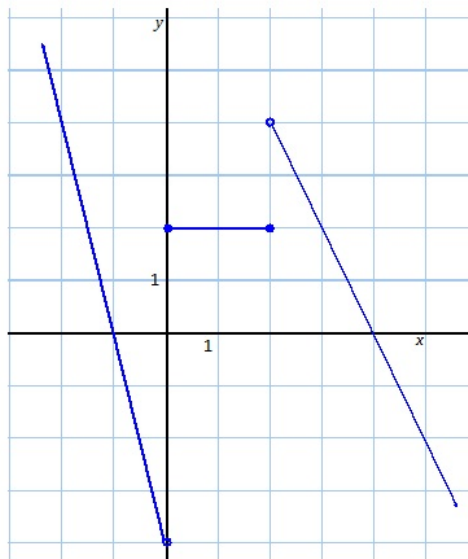
c.



Graph of k

$$k(x) = \begin{cases} x + 3 & x \leq -1 \\ 2 & -1 \leq x \leq 1 \\ x + 1 & x \geq 1 \end{cases}$$

d.



Graph of h

$$h(x) = \begin{cases} -4x - 4 & x < 0 \\ 2 & 0 \leq x \leq 2 \\ -2x + 8 & x > 2 \end{cases}$$