# Lesson 24: Piecewise and Step Functions in Context

## Student Outcomes

- Students create piecewise and step functions that relate to real-life situations and use those functions to solve problems.
- Students interpret graphs of piecewise and step functions in a real-life situation.

## Lesson Notes

Students study airport parking rates and consider making a change to them to raise revenue for the airport. They model the parking rates with piecewise and step functions and apply transformations and function evaluation skills to solve problems about this real-life situation. The current problem is based on the rates at the Albany International Airport (http://www.albanyairport.com/parking_rates.php).

Do not assume that just because this lesson is about piecewise linear functions that it will be easy for your students. Please read through and do all the calculations carefully before teaching this lesson. By doing the calculations you will get a sense of how much time this lesson will take. To finish this modeling lesson in one day you will need to break the class into four large groups (which may be split into smaller groups that work on the same task if you wish). Depending on your student population, you may wish to break this lesson into two days.

## Classwork

### Opening Exercise (2 minutes)

Introduce the lesson by presenting the following two scenarios. Model how to compute the parking costs for a 2.75-hour stay.

**Opening Exercise**

**Here are two different parking options in the city.**

| 1-2-3 Parking | Blue Line Parking |
|---|---|
| $\$6$ for the 1st hr (or part of an hr)<br>$\$5$ for the 2nd hr (or part of an hr)<br>$\$4$ for each hr (or part of an hr) starting with the 3rd hr | $\$5$ per hour up to $5$ hr<br>$\$4$ per hr after that |

**The cost of a $2.75$-hr stay at 1-2-3 Parking would be $\$6 + \$5 + \$4 = \$15$. The cost of a $2.75$-hr stay at Blue Line Parking would be $\$5(2.75) = \$13.75$.**

Students then use the rates at each parking garage to determine which one would cost less money if they planned to stay for exactly 5.25 hours in the Opening Exercise.

### Opening Exercise (3 minutes)

**Opening Exercise**

**Which garage costs less for a $5.25$-hour stay? Show your work to support your answer.**

***1-2-3 Parking:*** $\$27$. ***Blue Line Parking:*** $\$26$.

### Discussion (5 minutes)

Lead a discussion about the type of function that could be used to model the relationship between the length of the stay and the parking rates at each garage.

- What is this problem about?
  - *It is about comparing parking rates at two different garages.*
- What are the quantities in this situation?
  - *Time and money are two quantities in this situation.*
- What types of functions would model each parking plan?
  - *1-2-3 Parking would be modeled by a step function and Blue Line Parking by a piecewise function. For 1-2-3 Parking, the charge for a fraction of an hour is the same as the hourly rate so a step function would be a better model.*
- What would be the domain and what would be the range in each situation?
  - *A reasonable domain could be* $0$–$24$ *hours. The range would be the cost based on the domain. For 1-2-3, the range would be* $\{6, 11, 15, 19, 23, 27 \ldots 99\}$ *and for Blue Line, the range would be* $(0, 101]$.

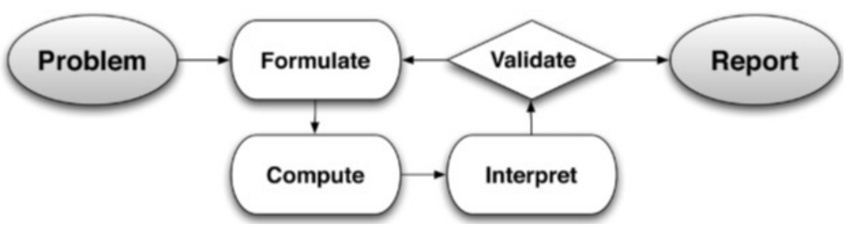
*Scaffolding:*

- Allow students to use technology throughout this lesson.
- Pay careful attention to assigning proper intervals for each piece. You may need to model this more closely for your classes.

Students will revisit this Opening Exercise in the Problem Set. Optionally, you may consider doing Problem 1 part (b) from the Problem Set for this lesson here.

#### Additional Lesson Notes

Transition to the Exploratory Challenge by announcing that students will spend the rest of this session working on a modeling problem. You can review the modeling cycle with your class if you wish to alert them to the distinct phases in the modeling cycle. You can activate thinking by using questions similar to the ones above to engage students in the airport parking problem.



### Exploratory Challenge (30 minutes)

In this portion of the lesson, students access parking rates at the Albany International Airport. They create algebraic models for the various parking structures and then analyze how much money is made on a typical day. Finally, they recommend how to adjust the rates to increase revenues by $10\%$. These parking rates are based upon rates that went into effect in 2008. Updated rates can be accessed by visiting the Albany International Airport website. The parking ticket data has been created for the purposes of this problem and is not based on information provided by the airport (although the data is based upon the average daily revenue generated using the figure below for total revenue in a year).

Assign each group one parking rate to model. These will all be step functions. Students will work in groups to create a function to model their assigned parking rate. Make sure each rate is assigned to at least one group. You may also encourage students to generate tables and graphs for each parking rate, both as a scaffold to help define the function algebraically and to create a richer model. Offering a variety of tools, such as graph paper, calculators, and graphing software will highlight MP.5.

If a group finishes problems 2–3 of this challenge early, it can repeat the questions for a different rate of its choice. If time permits, you could have groups graph their functions in the Cartesian plane. After all groups have completed their assigned rates, share results as a whole class. Give groups time enough to record the other models and their results for the revenue generated. Groups must then decide how to alter the price structure to get a $10\%$ increase in average daily revenue based on the data available. Explain to groups that in a real situation, the finance department would have access to revenue data for a longer period of time and would be better able to forecast and make recommendations regarding increases in revenue. In 2008, the Albany International Airport generated over $\$11{,}000{,}000$ in parking revenues.

**Exploratory Challenge**

**Helena works as a summer intern at the Albany International Airport. She is studying the parking rates and various parking options. Her department needs to raise parking revenues by $10\%$ to help address increased operating costs. The parking rates as of 2008 are displayed below. Your class will write piecewise linear functions to model each type of rate and then use those functions to develop a plan to increase parking revenues.**

Parking Rates (Effective October 28, 2008)

**Short Term Rates**

*Located on first floor of parking garage and front of the terminal*

| | |
|---|---|
| First Half Hour: | FREE |
| Second Half Hour: | $2.00 |
| Each Additional Half Hour: | $1.00 |
| Maximum Daily Rate: | $24.00 |

**Garage Parking Rates**

*Located on floors two, three, four and five of the parking garage*

| | |
|---|---|
| First Hour: | $2.00 |
| Each Additional Hour: | $2.00 |
| Maximum Daily Rate: | $12.00 |
| Five Consecutive Days: | $50.00 |
| Seven Consecutive Days: | $64.00 |

**Long Term Parking Rates**

*Located behind the parking garage*

| | |
|---|---|
| First Hour: | $2.00 |
| Each Additional Hour: | $1.00 |
| Maximum Daily Rate: | $9.00 |
| Five Days: | $36.00 |
| Seven Days: | $45.00 |

**Economy Remote Lot E - Shuttle to and from Terminal**

| | |
|---|---|
| First Hour: | $1.00 |
| Hourly Rate: | $1.00 |
| Maximum Daily Rate: | $5.00 |

Students will definitely have questions about how to interpret the different rates. Stress the second sentence of the problem statement. Let students discuss in their groups how to interpret the rates (it is part of the formulation of the problem in the modeling cycle), but gently guide them to adopting the following guidelines after that discussion:

**Short-Term Rates:** Since it is free for the first $\frac{1}{2}$ hour but \$2 for the second $\frac{1}{2}$ hour, students can use just one step function to model the first 12 hours, after which the parking fee is \$12 for the day. Suggest to students that it is not necessary to go past 24 hours—that is a rare occurrence and is usually dealt with on an ad hoc basis.

**Garage Rates/Long-Term Rates:** For this lesson, we will assume that the charge is \$50 for Garage and \$36 for Long-Term for either 5 or 6 days (do not prorate the time). Students may write a piecewise linear function for each day up to 7 days. This is acceptable, but challenge students to use a step function instead.

1. **Write a piecewise linear function using step functions that models your group's assigned parking rate. Note: Like in the Opening Exercise, assume that if the car is there for any part of the next time period, then that period is counted in full (i.e., $3.75$ hours is counted as $4$ hours, $3.5$ days is counted as $4$ days, etc.).**

   *Answers may vary.*

   *SHORT-TERM*

   $$S(x)=\begin{cases} 0 & 0\le x\le 0.5 \\ \lceil 2x\rceil & 0.5<x\le 12 \\ 24 & 12<x\le 24 \end{cases}$$

   *GARAGE*

   $$G(x)=\begin{cases} 2\lceil x\rceil & 0<x\le 6 \\ 12\left\lceil \frac{x}{24}\right\rceil & 6<x<120 \\ 50 & 120\le x<168 \\ 64 & 128\le x<192 \end{cases}$$

   *LONG-TERM*

   $$L(x)=\begin{cases} 2 & 0<x\le 1 \\ \lceil x\rceil+1 & 1<x\le 8 \\ 9\left\lceil \frac{x}{24}\right\rceil & 8<x<120 \\ 36 & 120\le x<168 \\ 45 & 168\le x\le 192 \end{cases}$$

   *ECONOMY*

   $$E(x)=\begin{cases} \lceil x\rceil & 0<x\le 5 \\ 5\left\lceil \frac{x}{24}\right\rceil & 5<x \end{cases}$$

**Helena collected all the parking tickets from one day during the summer to help her analyze ways to increase parking revenues and used that data to create the table shown below. The table displays the number of tickets turned in for each time and cost category at the four different parking lots.**

**Parking Tickets Collected on a Summer Day at the Albany International Airport**

| Short Term | | | Long Term | | | Parking Garage | | | Economy Remote | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Time on Ticket (hours) | Parking Cost ($) | Number of Tickets | Time on Ticket (hours) | Parking Cost ($) | Number of Tickets | Time on Ticket (hours) | Parking Cost ($) | Number of Tickets | Time on Ticket (hours) | Parking Cost ($) | Number of Tickets |
| 0.5 | 0 | 400 | 1 | 2 | 8 | 1 | 2 | 8 | 1 | 1 | |
| 1 | 2 | 600 | 2 | 3 | 20 | 2 | 4 | 12 | 2 | 2 | |
| 1.5 | 3 | 80 | 3 | 4 | 24 | 3 | 6 | 8 | 3 | 3 | |
| 2 | 4 | 64 | 4 | 5 | | 4 | 8 | 4 | 4 | 4 | |
| 2.5 | 5 | 8 | 5 | 6 | | 5 | 10 | 0 | 5 | 5 | |
| 3 | 6 | 24 | 6 | 7 | | 6 | 12 | 16 | 5 to 24 hrs | 5 | 84 |
| 3.5 | 7 | 4 | 7 | 8 | 60 | 6 to 24 | 12 | 156 | 2 days | 10 | 112 |
| 4 | 8 | | 8 | 9 | 92 | 2 days | 24 | 96 | 3 days | 15 | 64 |
| 4.5 | 9 | | 8 to 24 | 9 | 260 | 3 days | 36 | 40 | 4 days | 20 | 60 |
| 5 | 10 | | 2 days | 18 | 164 | 4 days | 48 | 12 | 5 days | 25 | 72 |
| 5.5 | 11 | | 3 days | 27 | 12 | 5-6 days | 50 | 8 | 6 days | 30 | 24 |
| 6 | 12 | | 4 days | 36 | 8 | 7 days | 64 | 4 | 7 days | 35 | 76 |
| 6.5 | 13 | | 5 days | 36 | 20 | | | | 8 days | 40 | 28 |
| 7 | 14 | | 6 days | 36 | 36 | | | | 9 days | 45 | 8 |
| 7.5 | 15 | | 7 days | 45 | 32 | | | | 10 days | 50 | 4 |
| 8 | 16 | 4 | | | | | | | 14 days | 70 | 8 |
| 8.5 | 17 | | | | | | | | 18 days | 90 | 4 |
| 9 | 18 | 8 | | | | | | | 21 days | 105 | 4 |
| 9.5 | 19 | | | | | | | | | | |
| 10 | 20 | | | | | | | | | | |
| 10.5 | 21 | | | | | | | | | | |
| 11 | 22 | | | | | | | | | | |
| 11.5 | 23 | | | | | | | | | | |
| 12 to 24 | 24 | 8 | | | | | | | | | |

For example, there were 600 short term 1-hr tickets charged $2 each. Total revenue for that type of ticket would be $1200.

Before moving groups on to tackle problems 3–4, lead a quick discussion around the data in the table. Ask the following questions. All of the answers to the discussion questions below are sample responses. Encourage a wide variety of reasonable responses from your students as long as they are realistic.

- What do the values in the number of tickets columns represent?
  - *The number of tickets turned in as cars left the parking area for various hours in the lots.*
- What does the last row of the economy lot table mean?
  - *It means that* $4$ *people left the parking lot after parking their cars there for* $21$ *days and each paid* $\$105$.
- How much do you think this data varies from day to day, month to month, etc.?
  - *It might vary day to day—for example, more people checking out on a Friday or fewer people traveling on the weekends. It might also vary season to season. Maybe people use the parking garage more in the winter months to avoid having their cars being covered with snow.*
- What if Helena collected this data on July 4? How would that information influence your thinking about whether or not this is a typical day's collection of parking fees?
  - *This would probably not be a typical day. Fewer people travel on holidays, so parking revenues might be less than usual.*

- What assumptions would you need to make to use this data to make a recommendation about raising yearly parking revenues by $10\%$?
  - *You would need to assume that on average, approximately $\$8700$ is collected each day and the distribution of tickets on any given day is similar to this one.*

2. **Compute the total revenue generated by your assigned rate using the following parking ticket data.**

   ***Total revenue for Short Term:*** $\$2,308$

   ***Total revenue for Long Term:*** $\$10,840$

   ***Total revenue for Parking Garage:*** $\$7,184$

   ***Total revenue for Economy:*** $\$11,900$

   ***Total revenue from all lots:*** $\$32,232$

3. **The Albany International Airport wants to increase average daily parking revenue by $10\%$. Make a recommendation to management of one or more parking rates to change to increase daily parking revenue by $10\%$. Then use the data Helena collected to show that revenue would increase by $10\%$ if they implement the recommended change.**

   ***A 10% increase would be a total of $\$35,455.20$. Student solutions will vary but should be supported with a calculation showing that their changes will result in a $10\%$ increase in parking revenue. The simplest solution would be to raise each rate by $10\%$ across the board. However, consumers may not like the strange-looking parking rates. Another proposal would be to raise short-term rates by $\$0.50$/half-hour and raise economy rates to $\$6$ per day instead of $\$5$.***

## Exit Ticket (5 minutes)

Students interpret a step function that represents postage costs. They create a piecewise linear function to model the data.

Name ___________________________________________________ Date____________________

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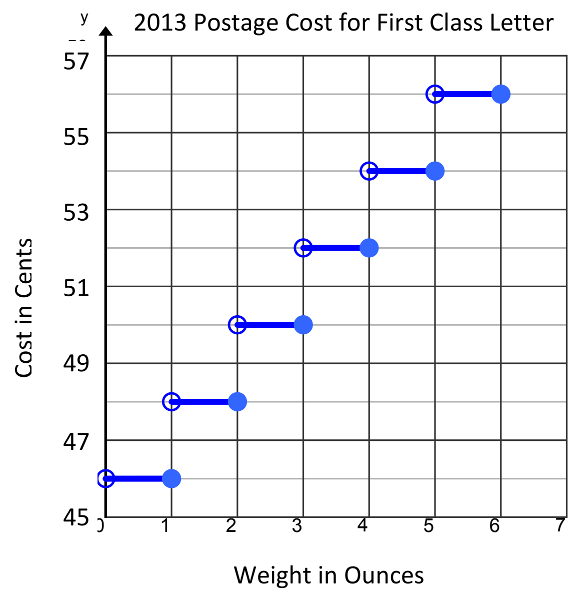
## Exit Ticket

1. Use the graph to complete the table.



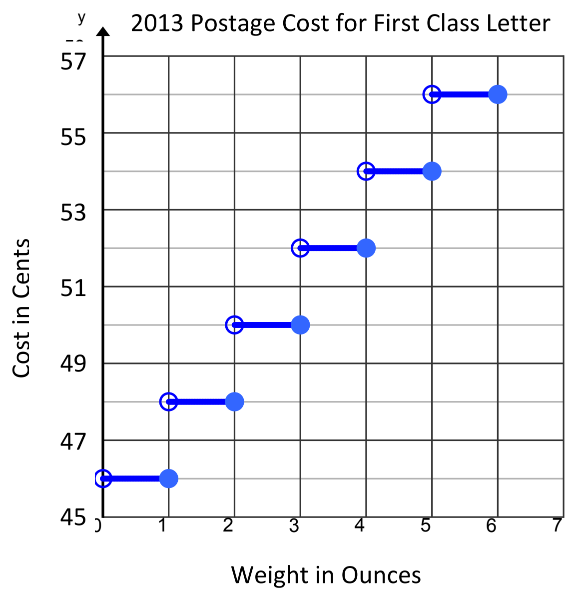

| **Weight in ounces, $x$** | $2$ | $2.2$ | $3$ | $3.5$ | $4$ |
|---|---|---|---|---|---|
| **Cost of postage, $C(x)$** | | | | | |

2. Write a formula involving step functions that represents the cost of postage based on the graph shown above.

3. If it cost Trina $\$0.54$ to mail her letter, how many ounces did it weigh?

## Exit Ticket Sample Solutions

1. **Use the graph to complete the table.**




| Weight in ounces, $x$ | $2$ | $2.2$ | $3$ | $3.5$ | $4$ |
|---|---|---|---|---|---|
| Cost of postage, $C(x)$ | $48$ | $50$ | $50$ | $52$ | $52$ |

2. **Write a formula involving step functions that represents the cost of postage based on the graph shown above.**

   $f(x) = 2\lceil x \rceil + 44, 0 < x \le 6$

3. **If it cost Trina $\$0.54$ to mail her letter, how many ounces did it weigh?**

   *It weighed more than $4$ oz. but less than or equal to $5$ oz.*

## Problem Set Sample Solutions

These problems provide a variety of contexts for using piecewise and step functions to model situations. The following solutions indicate an understanding of the objectives of this lesson:

1. **Recall the parking problem from the Opening Exercise.**

   a. **Write a piecewise linear function $P$ using step functions that models the cost of parking at 1-2-3 Parking for $x$ hrs.**

   $$P(x) = \begin{cases} 6\lceil x \rceil & 0 \le x \le 1 \\ 5\lceil x-1 \rceil + 6 & 1 < x \le 2 \\ 4\lceil x-2 \rceil + 11 & 2 < x \end{cases}$$

   b. **Write a piecewise linear function $B$ that models the cost of parking at Blue Line parking for $x$ hrs.**

   $$B(x) = \begin{cases} 5x & 0 \le x \le 5 \\ 4(x-5) + 25 & 5 < x \end{cases}$$

   c. **Evaluate each function at $2.75$ and $5.25$ hrs. Do your answers agree with the work in the Opening Exercise? If not, refine your model.**

   *$P(2.75) = 15$ and $B(2.75) = 13.75$,*
   *$P(5.25) = 27$ and $B(5.25) = 26$.*

   d. **Is there a time where both models have the same parking cost? Support your reasoning with graphs and/or equations.**

   *When $x = 5.5, 6.5, 7.5, \ldots$*

   e. **Apply your knowledge of transformations to write a new function that would represent the result of a $2 across the board increase in hourly rates at 1-2-3 Parking. (Hint: Draw its graph first and then use the graph to help you determine the step functions and domains.)**

   $$P_{new}(x) = \begin{cases} (2+6)\lceil x \rceil & 0 \le x \le 1 \\ (2+5)\lceil x-1 \rceil + 8 & 1 < x \le 2 \\ (2+4)\lceil x-2 \rceil + 15) & 2 < x \end{cases}$$

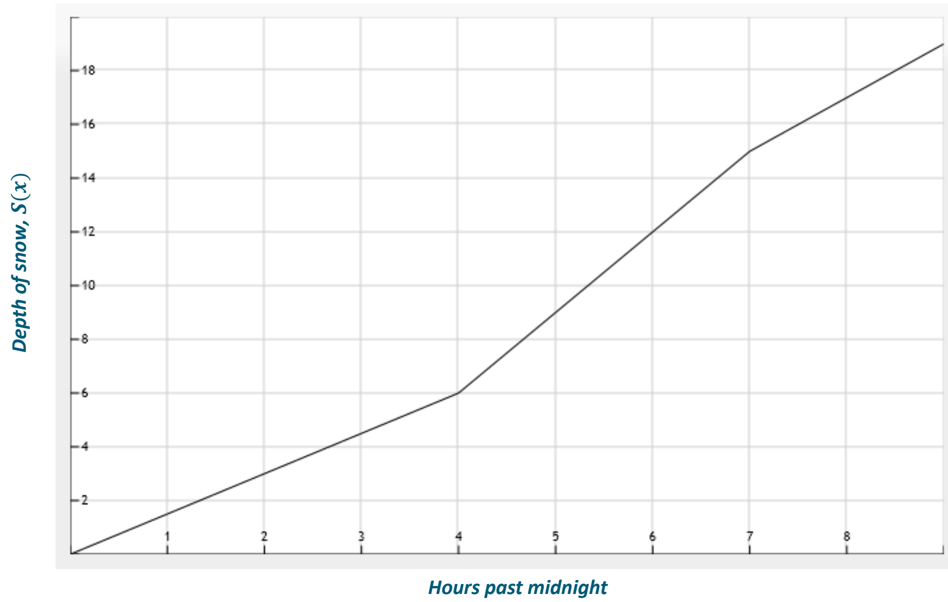
2. **There was no snow on the ground when it started falling at midnight at a constant rate of $1.5$ inches per hour. At 4:00 a.m., it starting falling at a constant rate of $3$ in. per hr., and then from 7:00 a.m. to 9:00 a.m., snow was falling at a constant rate of $2$ in. per hr. It stopped snowing at 9:00 a.m. (Note—this problem models snow falling by a constant rate during each time period. In reality, the snowfall rate might be very close to constant but is unlikely to be perfectly uniform throughout any given time period.)**

   a. **Write a piecewise linear function that models the depth of snow as a function of time since midnight.**

   *Let $S$ be a function that gives the depth of snow $S(x)$ on the ground $x$ hours after midnight.*

   $$S(x) = \begin{cases} 1.5x & 0 \le x < 4 \\ 3(x-4) + 6 & 4 \le x < 7 \\ 2(x-7) + 15 & 7 \le x \le 9 \end{cases}$$

b. Create a graph of the function.




c. When was depth of the snow on the ground $8$ inches deep?

*$S(x) = 8$ when $3(x-4)+6 = 8$. The solution of this equation is $x = \frac{14}{3}$ hours after midnight or at 4:40 a.m.*

d. How deep was the snow at $9:00$ a.m.?

*$S(9) = 19$ in.*

3. If you earned up to $\$113,700$ in $2013$ from an employer, your Social Security tax rate was $6.2\%$ of your income. If you earned over $\$113,700$, you pay a fixed amount of $\$7,049.40$.

a. Write a piecewise linear function to represent the $2013$ Social Security taxes for incomes between $\$0$ and $\$500,000$.

*Let* $f(x) = \begin{cases} 0.062x, & 0 < x \le 113,700 \\ 7,049.40, & 113,700 < x \le 500,000 \end{cases}$

*where $x$ is income in dollars and $f(x)$ is the $2013$ Social Security tax.*

b. How much Social Security tax would someone who made $\$50,000$ owe?

*$f(50,000) = 3,100$. The person would owe $\$3,100$.*

c. How much money would you have made if you paid $\$4000$ in Social Security tax in $2013$.

*$f(x) = 4000$ when $x = 64,516.129$. You would have made $\$64,516.13$.*

d. What is the meaning of $f(150,000)$? What is the value of $f(150,000)$?

*The amount of Social Security tax you would owe if you earned $\$150,000$. $f(150,000) = \$7,049.40$.*

4. The function $f$ gives the cost to ship $x$ lbs. via Fed Ex Standard Overnight Rates to Zone 2 in 2013.

$$f(x) = \begin{cases} 21.50 & 0 < x \leq 1 \\ 23.00 & 1 < x \leq 2 \\ 24.70 & 2 < x \leq 3 \\ 26.60 & 3 < x \leq 4 \\ 27.05 & 4 < x \leq 5 \\ 28.60 & 5 < x \leq 6 \\ 29.50 & 6 < x \leq 7 \\ 31.00 & 7 < x \leq 8 \\ 32.25 & 8 < x \leq 9 \end{cases}$$

a. How much would it cost to ship a 3 lb. package?

$f(3) = 24.7$, *The cost is* $\$24.70$.

b. How much would it cost to ship a $7.25$ lb. package.

$f(7.25) = 31$, *The cost is* $\$31.00$.

c. What is the domain and range of $f$?

*Domain:* $x \in (0, 9]$, *Range:* $f(x) \in \{21.5, 23, 24.7, 26.6, 27.05, 28.6, 29.5, 31, 32.25\}$

d. Could you use the ceiling function to write this function more concisely? Explain your reasoning.

*No. The range values on the ceiling function differ by a constant amount. The rates in function $f$ do not increase at a constant rate.*

5. Use the floor or ceiling function and your knowledge of transformations to write a piecewise linear function $f$ whose graph is shown below.

$f(x) = -\lceil x \rceil + 3$ *or* $f(x) = \lfloor -x \rfloor + 4$