Lesson 16: Graphs Can Solve Equations Too

Student Outcomes

- Students discover that the multi-step and exact way of solving $|2x-5|=|3x+1|$ using algebra can sometimes be avoided by recognizing that an equation of the form $f(x)=g(x)$ can be solved visually by looking for the intersection points of the graphs of $y=f(x)$ and $y=g(x)$.

Lesson Notes

This lesson focuses on A.REI.11 which emphasizes that the x-coordinates of the intersection points of the graphs of two functions f and g are the solutions to the equation $f(x)=g(x)$. This lesson ties work from Module 1 on solving systems of two-variable equations to work with functions and leads students to the understanding of what the solution set to a one-variable equation can be.

Classwork

Opening Exercises 1–4 (5 minutes)

In the opening, instruct students to solve for x in the equation by isolating the absolute value expression and separating the solution into two cases: one for the absolute value expression that represents distance from 0 in the positive direction of the number line, and one for the distance from 0 in the negative direction. In Opening Exercise 2, we introduce the functions f and g somewhat artificially and consider the graphs $y=f(x)$ and $y=g(x)$. Students quickly recognize that this series of artificial moves actually has a solid purpose: to solve for x visually using the graphs of functions. After the labor required in Opening Exercise 1, students should appreciate this clever way of solving for x.

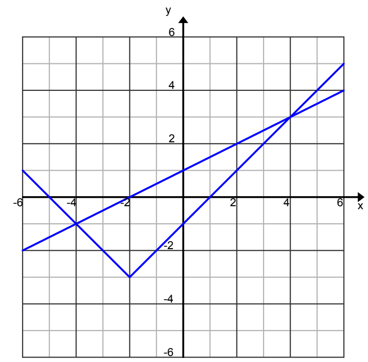
Opening Exercises

1. **Solve for x in the following equation:** $|x+2|-3=0.5x+1$

$$|x+2|=0.5x+4$$

$$x+2=0.5x+4 \qquad \textit{or} \qquad x+2=-(0.5x+4)$$
$$0.5x=2 \qquad\qquad 1.5x=-6$$
$$x=4 \qquad\qquad x=-4$$

2. Now let $f(x) = |x+2| - 3$ and $g(x) = 0.5x + 1$. When does $f(x) = g(x)$? To answer this, first graph $y = f(x)$ and $y = g(x)$ on the same set of axes.



3. When does $f(x) = g(x)$? What is the visual significance of the points where $f(x) = g(x)$?

$f(x) = g(x)$ when $x = 4$ and when $x = -4$; $(4, 3)$ and $(-4, -1)$. The points where $f(x) = g(x)$ are the intersections of the graphs of f and g.

4. Is each intersection point (x, y) an element of the graph f and an element of the graph of g? In other words, do the functions f and g really have the same value when $x = 4$? What about when $x = -4$?

Yes. You can determine this by substituting $x = -4$ and $x = 4$ into both f and g.

$$-1 = |-4+2| - 3 \qquad -1 = 0.5(-4) + 1$$
$$-1 = 2 - 3 \qquad -1 = -1$$
$$-1 = -1$$

$$3 = |4+2| - 3 \qquad 3 = 0.5(4) + 1$$
$$3 = 6 - 3 \qquad 3 = 3$$
$$3 = 3$$

$f(x)$ and $g(x)$ have the same value at each x.

Be sure to review the solutions to these problems with the entire class before moving on. Before sharing as a whole group, give students time to compare their answers with a partner.

Discussion (8 minutes)

Lead a discussion that ties together the work in the Opening Exercises. The idea here is to create an equation $f(x) = g(x)$ and show that the x-coordinates of the intersection points are the solution set to this equation. This example will draw on MP.7 as students will need to look closely to determine the connection between the functions and equations involved.

First summarize what we know from the Opening Exercise. Ask students to discuss this with a partner and then call on a few people to share their thoughts. Make sure the following point is clear to the class:

- For functions f and g, we have found special ordered pairs that (1) are points in the intersection of the graph of f and the graph of g, (2) are solutions to a system of equations given by the equations $y = f(x)$ and $y = g(x)$, and (3) have x-values that satisfy the equation $f(x) = g(x)$.

Write this equation on the board. Ask students to use the solutions from the Opening Exercise to answer the following:

- Are the x-coordinates of the intersection points solutions to this equation?
  - *Yes, when I substituted the x-coordinates into the equation I got a true number sentence.*
- Are the y-coordinates of the intersection points solutions to this equation?
  - *No, when I substituted the y-coordinates into the equation I got a false number sentence.*

- Do you think there are any other solutions to this equation? How could you be sure?
  - *I do not think so because each side of the equation is one of the functions shown in the graphs. The shape of the graph makes me think that there are no other intersection points. We could algebraically solve the equation to prove that these are the only solutions.*

Give students (in groups of three or four) time to debate the next discussion question. Have different students share their thinking with the whole class.

- Is it always true that the x-coordinates of the intersection points of the graphs of two functions will be the solution set to the equation $f(x) = g(x)$?
  - *Yes. To create the graphs of f and g we cycle through some of the domain values x and plot the pairs $\big(x, f(x)\big)$ and $\big(x, g(x)\big)$. The points that these two functions have in common will have x-values that satisfy the equation $f(x) = g(x)$ because this equation asks us to find the domain elements x that make the range elements $f(x)$ and $g(x)$ equal.*
- What is the advantage of solving an equation graphically by finding intersection points in this manner?
  - *It can be helpful when the equations are complicated or impossible to solve algebraically. It will also be useful when estimating solutions is enough to solve a problem. The graphically estimated solutions might give insight into ways to solve the equation algebraically.*

Example 1 (8 minutes)

This example provides an opportunity to model explicitly how to use graphs of functions to solve an equation. As you work with students, guide them to label the graphs similarly to what is shown in the solutions below. This will reinforce proper vocabulary. In this guided example, students will complete the graphs of the functions and then fill in the blanks as you discuss as a whole class. This example should reinforce the previous discussion.

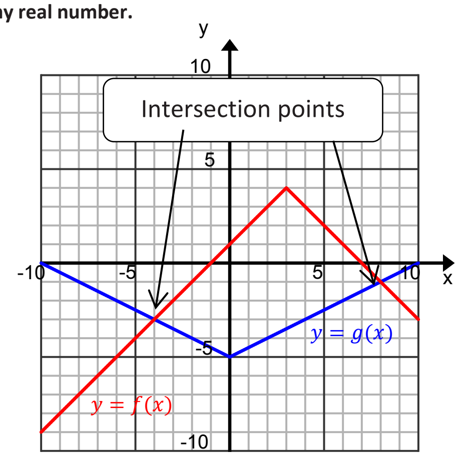
Example 1

Solve this equation by graphing two functions on the same Cartesian plane: $|0.5x| - 5 = -|x - 3| + 4$.

Let $f(x) = |0.5x| - 5$ and let $g(x) = -|x - 3| + 4$ where x can be any real number.

We are looking for values of x at which the functions f and g have the same output value.

Therefore, we set $y = f(x)$ and $y = g(x)$ so we can plot the graphs on the same coordinate plane:



From the graph, we see that the two intersection points are

_____________ and _____________.

$(-4, -3)$ and $(8, -1)$

> **The fact that the graphs of the functions meet at these two points means that when x is ______ both $f(x)$ and $g(x)$ are ______, or when x is ______ both $f(x)$ and $g(x)$ are ______.**
>
> $-4, -3, 8, -1$
>
> **Thus, the expressions $|0.5x| - 5$ and $-|x-3| + 4$ are equal when x = ______ or when x = ______.**
>
> $-4,\ 8$
>
> **Therefore, the solution set to the original equation is ______.**
>
> $\{-4, 8\}$

After working with the class to use their knowledge from the previous lesson to create these graphs, lead a discussion that emphasizes the following:

- We are looking for values of x where the values $f(x)$ and $g(x)$ are the same. In other words, we want to identify the points $(x, f(x))$ of the graph of f and the points $(x, g(x))$ of the graph of g that are the same. This will occur where the graphs of the two functions intersect.
- We must also convince ourselves that these are the only two solutions to this equation. Pose the question: How can we be certain that these two intersection points are the only two solutions to this equation? Give students time to discuss this with a partner or in a small group. Encourage them to reason from the graphs of the functions, rather than solving the equation.
  - *For all $x < -4$ or $x > 8$, the differences in the y-values of two functions are always greater than zero. To see this, note that $f(x) - g(x) = |0.5x| - 5 - (-|x-3| + 4) = |0.5x| + |x-3| - 9$. The last expression is greater than zero when $|0.5x| + |x-3| > 9$, which is certainly true for $x < -4$ or $x > 8$ (by inspection—there is no need to solve this equation due to the graph). Hence, the only solutions can occur in the interval $-4 \le x \le 8$, of which there are two.*
- If time permits, challenge students to experiment with sketching in the same Cartesian plane the graphs of two functions (each of which involve taking an absolute value) that intersect at zero points, exactly one point, exactly two points, or an infinite number of points.

Example 2 (10 minutes)

This example requires graphing calculators or other graphing software that is capable of finding the intersection points of two graphs. As you work through this example, discuss and model how to:

- Enter functions into the graphing tool, graph them in an appropriate viewing window to see the intersection points, and use the features of the graphing technology to determine the coordinates of the intersection points.
- Show the difference between 'tracing' to the intersection point and using any built-in functions that determine the intersection point.
- Have students estimate the solutions from the graph before using the built-in features.
- Have students verify that the x-coordinates of the intersecting points are solutions to the equations.
- Have students sketch the graphs and label the coordinates of the intersection points on their handouts.

> *Scaffolding:*
>
> Refer to video lessons on the Internet for further examples and support for teaching this process using technology.

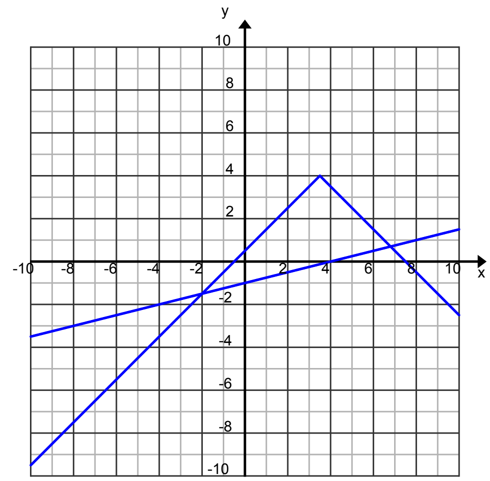
Example 2

Solve this equation graphically: $-|x-3.5|+4=0.25x-1$

a. **Write the two functions represented by each side of the equation.**

Let $f(x)=-|x-3.5|+4$ *and let* $g(x)=0.25x-1$*, where* x *can be any real number.*

b. **Graph the functions in an appropriate viewing window.**



c. **Determine the intersection points of the two functions.**

$(-2,-1.5)$ *and* $(6.8,0.7)$

d. **Verify that the** x**-coordinates of the intersection points are solutions to the equation.**

Let $x=-2$ *then*

$$-|-2-3.5|+4=0.25(-2)-1$$
$$-5.5+4=-0.5-1$$
$$-1.5=-1.5$$

Let $x=6.8$ *then*

$$-|6.8-3.5|+4=0.25(6.8)-1$$
$$-3.3+4=1.7-1$$
$$0.7=0.7$$

Exercises 1–5 (8 minutes)

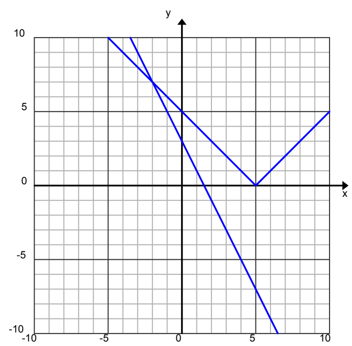
Students practice using graphs of functions to solve equations. Students should work through these exercises in small groups and discuss their solutions as they work. Circulate among groups providing assistance as needed.

Exercises 1–5

Use graphs to find approximate values of the solution set for each equation. Use technology to support your work. Explain how each of your solutions relates to the graph. Check your solutions using the equation.

1. $3 - 2x = |x - 5|$

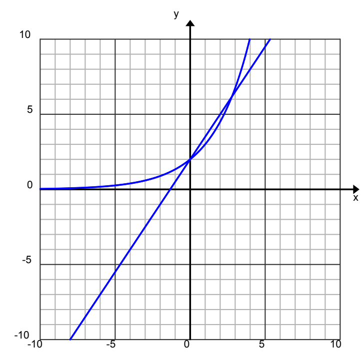
$x = -2$, the intersection point is $(-2, 7)$.



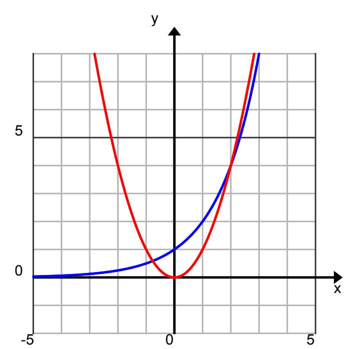
2. $2(1.5)^x = 2 + 1.5x$

First solution is $x = 0$, from the point $(0, 2)$;

Second solution answers will vary, x is about 2.7 or 2.8, based on the actual intersection point of $(2.776, 6.164)$.



3. **The graphs of the functions f and g are shown.**



a. **Use the graph to *approximate* the solution(s) to the equation $f(x) = g(x)$.**

Based on the graphs, the approximate solutions are $\{-0.7, 2\}$.

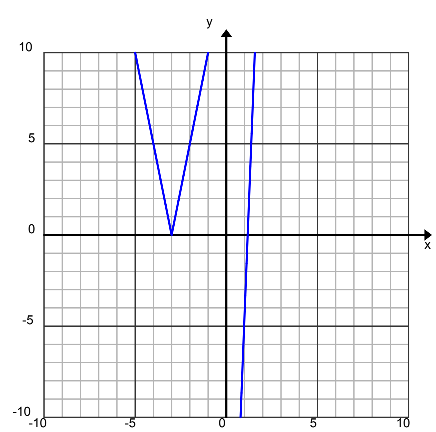
b. **Let $f(x) = x^2$ and let $g(x) = 2^x$. Find *all* solutions to the equation $f(x) = g(x)$. Verify any exact solutions that you determine using the definitions of f and g. Explain how you arrived at your solutions.**

By guessing and checking, $x = 4$ is also a solution of the equation because $f(4) = 16$ and $g(4) = 16$. Since the graph of the exponential function is increasing and increases more rapidly than the squaring function, there will only be 3 solutions to this equation. The exact solutions are $x = 2$ and $x = 4$ and an approximate solution is $x = -0.7$.

4. **The graphs of f, a function that involves taking an absolute value, and g, a linear function, are shown to the right. Both functions are defined over all real values for x. Tami concluded that the equation $f(x) = g(x)$ has no solution.**

Do you agree or disagree? Explain your reasoning.

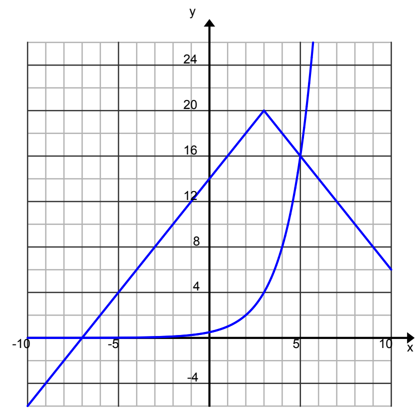
I disagree with Tami because we cannot see enough of this graph. The graph of the function shown to the left has a slope of 5. The graph of the function shown to the right has a slope greater than 5. Therefore, these two functions will intersect somewhere in the first quadrant. We would have to 'zoom out' to see the intersection point.



5. **The graphs of f, a function that involves taking the absolute value, and g, an exponential function, are shown below. Sharon said the solution set to the equation $f(x) = g(x)$ is exactly $\{-7, 5\}$.**

   Do you agree or disagree with Sharon? Explain your reasoning.

   I disagree with Sharon. We could say that the solution set is approximately $\{-7, 5\}$ but without having the actual equations or formulas for the two functions, we cannot be sure the x-values of the intersection points are exactly -7 and 5.



Closing (2 minutes)

In the last two exercises, students reflect on the limitations of solving an equation graphically. Debrief these exercises as a whole class and encourage different groups to present their reasoning to the entire class. Clarify any misconceptions before moving on, and give students time to revise their work. In Exercise 4, it is clear that there is an intersection point that is not visible in the viewing window provided. In Exercise 5, the intersection points would need to be estimated. If we do not have the exact algebraic solutions of the equation, then we can only estimate the solution set using graphs.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 16: Graphs Can Solve Equations Too

Exit Ticket

1. How do intersection points of the graphs of two functions f and g relate to the solution of an equation in the form $f(x) = g(x)$?

2. What are some benefits of solving equations graphically? What are some limitations?

Exit Ticket Sample Solutions

1. How do intersection points of the graphs of two functions f and g relate to the solution of an equation in the form $f(x) = g(x)$?

   The x-coordinates of the intersection points of the graphs of two functions are the solutions of the equation.

2. What are some benefits of solving equations graphically? What are some limitations?

   Benefits: Solving equations graphically can be helpful when you don't know how to solve the equation algebraically. It can also save you some time if you have technology available. This method can only provide approximate solutions, which may be all you need. Or the approximate solutions may give you insight into how to solve the equation algebraically.

   Limitations: You cannot be sure you have found all the solutions to an equation unless you can reason about the graphs of the functions themselves and convince yourself that no other intersection points are possible. The solutions found graphically rely on eyeballing. There is no guarantee that they are exact solutions; sometimes they are, but other times they are just decent approximations.

Problem Set Sample Solutions

1. Solve the following equations graphically. Verify the solution sets using the original equations.

   a. $2x - 4 = \sqrt{x + 5}$

      Approximately 3.4538

   b. $|x| = x^2$

      $\{-1, 0, 1\}$

   c. $x + 2 = x^3 - 2x - 4$

      Approximately 2.3553

   d. $|3x - 4| = 5 - |x - 2|$

      $\{0.25, 2.75\}$

   e. $0.5x^3 - 4 = 3x + 1$

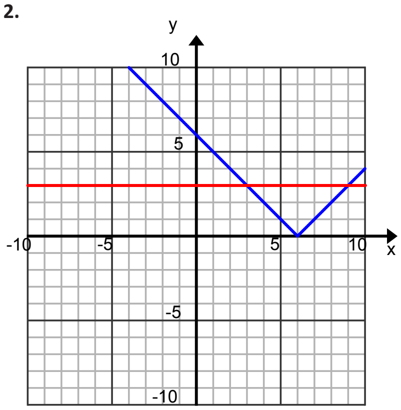
      Approximately 3.0467

   f. $6\left(\frac{1}{2}\right)^{5x} = 10 - 6x$

      Approximately -0.1765 *and* 1.6636

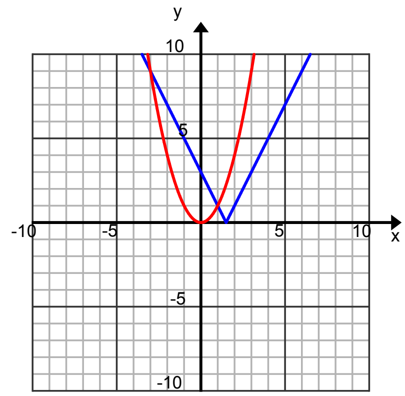
In each problem, the graphs of the functions f and g are shown on the same Cartesian plane. Estimate the solution set to the equation $f(x) = g(x)$. Assume that the graphs of the two functions only intersect at the points shown on the graph.

2.



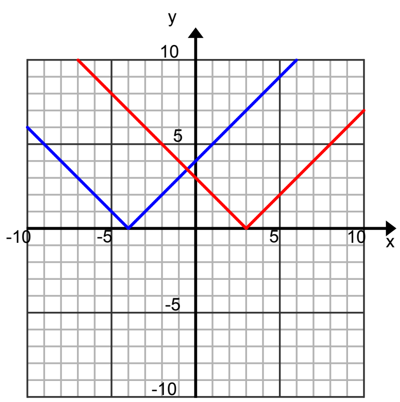
$\{3, 9\}$

3.



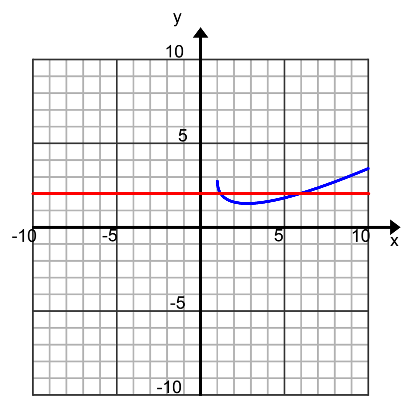
$\{-3, 1\}$

4.



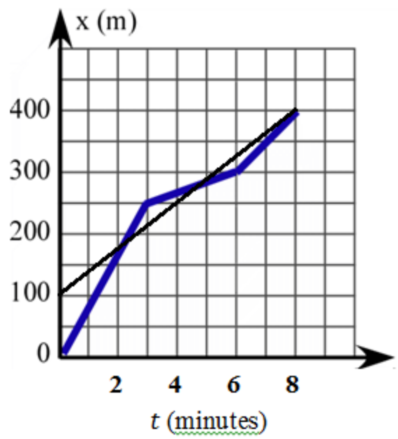
$\left\{\frac{1}{2}\right\}$

5.



$\{1.2, 6\}$

6. The graph below shows Glenn's distance from home as he rode his bicycle to school, which is just down his street. His next-door neighbor Pablo, who lives 100 m closer to the school, leaves his house at the same time as Glenn. He walks at a constant velocity, and they both arrive at school at the same time.



a. Graph a linear function that represents Pablo's distance from Glenn's home as a function of time.

b. Estimate when the two boys pass each other.

They cross paths at about 2 minutes and 5 minutes. I can tell that by finding the x-coordinates of the intersection points of the graphs of the functions.

c. Write piecewise-linear functions to represent each boy's distance and use them to verify your answer to part (b).

$$P(t) = 100 + 37.5t$$

$$G(t) = \begin{cases} \frac{250}{3}t & 0 \le t \le 3 \\ 200 + \frac{50}{3}t & 3 < t \le 6 \\ 50t & 6 < t \le 8 \end{cases}$$

At about 2 minutes: $100 + \frac{75}{2}t = \frac{250}{3}t$ *or* $600 + 225t = 500t$ *or* $275t = 600$ *or* $t = \frac{24}{11}$ *min.*

At about 5 minutes: $100 + \frac{75}{2}t = 200 + \frac{50}{3}t$ *or* $225t = 600 + 100t$ *or* $125t = 600$ *or* $t = \frac{24}{5} = 4.8$ *min.*