Lesson 6: Finite and Infinite Decimals

Student Outcomes

- Students know that every number has a decimal expansion (i.e., is equal to a finite or infinite decimal).
- Students know that when a fraction has a denominator that is the product of 2's and/or 5's, it has a finite decimal expansion because the fraction can then be written in an equivalent form with a denominator that is a power of 10.

Lesson Notes

The terms *expanded form of a decimal* and *decimal expansion* are used throughout this topic. The expanded form of a decimal refers to the value of a number written as a sum. For example, the expanded form of the decimal 0.125 is $\frac{1}{10}+\frac{2}{10^2}+\frac{5}{10^3}$. When students are asked to determine the decimal expansion of a number like $\sqrt{2}$ we expect them to write the decimal value of the number. For example, the decimal expansion of $\sqrt{2}$ is approximately 1.4142. The examination of the decimal expansion leads to an understanding of irrational numbers. Numbers with decimal expansions that are infinite (i.e., non-terminating) and do not have a repeat block are called irrational numbers. Numbers with finite (i.e., terminating) decimal expansions, as well as those numbers that are infinite with repeat blocks, are called rational numbers. Students will be exposed to the concepts of finite and infinite decimals here; however, the concept of irrational numbers will not be formally introduced until Lesson 11.

Classwork

Opening Exercises 1–5 (7 minutes)

Provide students time to work, then share their responses to Exercise 5 with the class.

Opening Exercises 1–5

1. **Use long division to determine the decimal expansion of $\frac{54}{20}$.**

   The number $\frac{54}{20}=2.7$.

2. **Use long division to determine the decimal expansion of $\frac{7}{8}$.**

   The number $\frac{7}{8}=0.875$.

3. **Use long division to determine the decimal expansion of $\frac{8}{9}$.**

   The number $\frac{8}{9}=0.8888\ldots$

4. **Use long division to determine the decimal expansion of $\frac{22}{7}$.**

   The number $\frac{22}{7}=3.142857\ldots$

MP.2

5. **What do you notice about the decimal expansions of Exercises 1 and 2 compared to the decimal expansions of Exercises 3 and 4?**

The decimal expansions of Exercises 1 and 2 ended. That is, when I did the long division I was able to stop after a few steps. That was different than the work I had to do in Exercises 3 and 4. In Exercise 3, I noticed that the same number kept coming up in the steps of the division, but it kept going on. In Exercise 4, when I did the long division it did not end. I stopped dividing after I found a few decimal digits of the decimal expansion.

Discussion (5 minutes)

Use the discussion below to elicit a dialog about finite and infinite decimals that may not have come up in the debrief of the Opening Exercises and to prepare students for what is covered in this lesson in particular (i.e., writing fractions as finite decimals without using long division).

Scaffolding:

Students may benefit from a graphic organizer that shows the key information regarding finite and infinite decimals. For example, the chart shown below:

Finite Decimals	Infinite Decimals
Definition: Examples:	Definition: Examples:

- Every number has a decimal expansion. That is, every number is equal to a decimal. For example, the numbers $\sqrt{3}$ and $\frac{17}{125}$ have decimal expansions. The decimal expansion of $\sqrt{3}$ will be covered in a later lesson. For now, we will focus on the decimal expansion of a number like $\frac{17}{125}$ and whether it can be expressed as a finite or infinite decimal.
- How would you classify the decimal expansions of Exercises 1–4?
  - *Exercises 1 and 2 are finite decimals and Exercises 3 and 4 are infinite decimals.*
- In the context of fractions, a decimal is, by definition, a fraction with a denominator equal to a power of 10. These decimals are known as finite decimals. The distinction must be made because we will soon be working with infinite decimals. Can you think of any numbers that are infinite decimals?
  - *Decimals that repeat or a number like pi are infinite decimals.*
- Decimals that repeat, such as $0.8888888\ldots$ or $0.454545454545\ldots$, are infinite decimals and typically abbreviated as $0.\overline{8}$ and $0.\overline{45}$, respectively. The notation indicates that the digit 8 repeats indefinitely and that the two-digit block 45 repeats indefinitely. The number π is also a famous infinite decimal: $3.1415926535\ldots$, which does not have a block of digits that repeats indefinitely.
- In Grade 7 you learned a general procedure for writing the decimal expansion of a fraction such as $\frac{5}{14}$ using long division. In the next lesson, we will closely examine the long division algorithm and why the procedure makes sense.
- Today, we will learn a method for converting a fraction to a decimal that does not require long division. Each of the fractions in the Examples and Exercises in *this* lesson are simplified fractions. The method we will learn requires that we begin with a simplified fraction.

MP.7

- Return to the Opening Exercise. We know that the decimals in Exercises 1 and 2 are finite, while the decimals in Exercises 3 and 4 are not. What do you notice about the denominators of these fractions that might explain this?
  - *The denominators of the fractions in Exercises 1 and 2 are the products of 2's and 5's. For example, the denominator $20 = 2 \times 2 \times 5$ and the denominator $8 = 2 \times 2 \times 2$. The denominators of the fractions in Exercises 3 and 4 were not the product of 2's and 5's. For example, $9 = 3 \times 3$ and $7 = 1 \times 7$.*

- Certain fractions, those whose denominators are a product of 2's or 5's or both, are equal to finite decimals. Fractions like $\frac{1}{4}$, $\frac{6}{125}$, and $\frac{9}{10}$ can be expressed as finite decimals because $4 = 2^2$, $125 = 5^3$, and $10 = 2 \times 5$.
- Other fractions like $\frac{5}{14}$, cannot be expressed as a finite decimal because $14 = 2 \times 7$. Therefore, $\frac{5}{14}$ has an infinite decimal expansion.

Example 1 (4 minutes)

Example 1

Consider the fraction $\frac{5}{8}$. Is it equal to a finite decimal? How do you know?

- Consider the fraction $\frac{5}{8}$. Is it equal to a finite decimal? How do you know?
  - *The fraction $\frac{5}{8}$ is equal to a finite decimal because the denominator 8 is a product of 2's. Specifically, $8 = 2^3$.*
- Since we know that the fraction $\frac{5}{8}$ is equal to a finite decimal, then we can find a fraction $\frac{k}{10^n}$, where k and n are positive integers, that will give us the decimal value that $\frac{5}{8}$ is equal to.
- We must find positive integers k and n, so that $\frac{5}{8} = \frac{k}{10^n}$.
- Explain the meaning of k and 10^n in the equation above.
  - *The number k will be the numerator, a positive integer, of a fraction equivalent to $\frac{5}{8}$ that has a denominator that is a power of 10, e.g., 10^2, 10^5, 10^n.*
- Recall what we learned about the laws of exponents in Module 1: $(ab)^n = a^n b^n$. We will now put that knowledge to use.

  We know that $8 = 2^3$ and $10^n = (2 \times 5)^n = 2^n \times 5^n$. Comparing the denominators of the fractions, $2^3 \times 5^n = 2^n \times 5^n = 10^n$.

  What must n equal?
  - *n must be 3.*
- To rewrite the fraction $\frac{5}{8}$ so that it has a denominator of the form 10^n, we must multiply 2^3 by 5^3. Based on what you know about equivalent fractions, by what must we multiply the numerator of $\frac{5}{8}$?
  - *To make an equivalent fraction we will need to multiply the numerator by 5^3 also.*
- By equivalent fractions:

$$\frac{5}{8} = \frac{5}{2^3} = \frac{5 \times 5^3}{2^3 \times 5^3} = \frac{5^4}{(2 \times 5)^3} = \frac{625}{10^3},$$

  where $k = 625$ and $n = 3$, both positive integers.
- Using the fraction $\frac{625}{10^3}$, we can write the decimal value of $\frac{5}{8}$. What is it? Explain.
  - *$\frac{5}{8} = 0.625$ because $\frac{625}{10^3} = \frac{625}{1000}$. Using what we know about place value we have six hundred twenty five thousandths, or 0.625.*

Example 2 (4 minutes)

Example 2

Consider the fraction $\frac{17}{125}$. Is it equal to a finite or infinite decimal? How do you know?

- Let's consider the fraction $\frac{17}{125}$ mentioned earlier. We want the decimal value of this number. Is it a finite or infinite decimal? How do you know?
  - *We know that the fraction $\frac{17}{125}$ is equal to a finite decimal because the denominator 125 is a product of 5's. Specifically, $125 = 5^3$.*
- What will we need to multiply 5^3 by so that is it equal to $(2 \times 5)^n = 10^n$?
  - *We will need to multiply by 2^3 so that $2^3 \times 5^3 = (2 \times 5)^3 = 10^3$.*
- Begin with $\frac{17}{125} = \frac{17}{5^3}$. Use what you know about equivalent fractions to rewrite $\frac{17}{125} = \frac{k}{10^n}$, and then the decimal form of the fraction.
  - $\frac{17}{125} = \frac{17}{5^3} = \frac{17 \times 2^3}{2^3 \times 5^3} = \frac{17 \times 8}{(2 \times 5)^3} = \frac{136}{10^3} = 0.136$

Exercises 6–10 (5 minutes)

Students complete Exercises 6–10 independently.

Exercises 6–10

Show your steps, but use a calculator for the multiplications.

6. **Convert the fraction $\frac{7}{8}$ to a decimal.**

   a. **Write the denominator as a product of 2's or 5's. Explain why this way of rewriting the denominator helps to find the decimal representation of $\frac{7}{8}$.**

   The denominator $8 = 2^3$. It is helpful to know that $8 = 2^3$ because it shows how many factors of 5 will be needed to multiply the numerator and denominator by so that an equivalent fraction is produced with a denominator that is a multiple of 10. When the denominator is a multiple of 10 the fraction can easily be written as a decimal using what I know about place value.

   b. **Find the decimal representation of $\frac{7}{8}$. Explain why your answer is reasonable.**

   $$\frac{7}{8} = \frac{7}{2^3} = \frac{7 \times 5^3}{2^3 \times 5^3} = \frac{875}{10^3} = 0.875$$

   The answer is reasonable because the decimal value, 0.875 is less than one just like the fraction $\frac{7}{8}$. Also, it is reasonable and correct because the fraction $\frac{875}{1{,}000} = \frac{7}{8}$; therefore, it has the decimal expansion 0.875.

7. **Convert the fraction $\frac{43}{64}$ to a decimal.**

The denominator $64 = 2^6$.

$$\frac{43}{64} = \frac{43}{2^6} = \frac{43 \times 5^6}{2^6 \times 5^6} = \frac{671875}{10^6} = 0.671875$$

8. **Convert the fraction $\frac{29}{125}$ to a decimal.**

The denominator $125 = 5^3$.

$$\frac{29}{125} = \frac{29}{5^3} = \frac{29 \times 2^3}{5^3 \times 2^3} = \frac{232}{10^3} = 0.232$$

9. **Convert the fraction $\frac{19}{34}$ to a decimal.**

Using long division, $\frac{19}{34} = 0.5588235 \ldots$

10. **Identify the type of decimal expansion for each of the numbers in Exercises 6–9 as finite or infinite. Explain why their decimal expansion is such.**

We know that the number $\frac{7}{8}$ had a finite decimal expansion because the denominator 8 is a product of 2's. We know that the number $\frac{43}{64}$ had a finite decimal expansion because the denominator 64 is a product of 2's. We know that the number $\frac{29}{125}$ had a finite decimal expansion because the denominator 125 is a product of 5's. We know that the number $\frac{19}{34}$ had an infinite decimal expansion because the denominator was not a product of 2's or 5's, it had a factor of 17.

Example 3 (4 minutes)

Example 3

Write $\frac{7}{80}$ as a decimal. Will it be finite or infinite? Explain.

- Let's write $\frac{7}{80}$ as a decimal. Will it be finite or infinite? Explain.
  - *We know that the fraction $\frac{7}{80}$ is equal to a finite decimal because the denominator 80 is a product of 2's and 5's. Specifically, $80 = 2^4 \times 5$.*
- What will we need to multiply $2^4 \times 5$ by so that is it equal to $(2 \times 5)^n = 10^n$?
  - *We will need to multiply by 5^3 so that $2^4 \times 5^4 = (2 \times 5)^4 = 10^4$.*
- Begin with $\frac{7}{80} = \frac{7}{2^4 \times 5}$, use what you know about equivalent fractions to rewrite $\frac{7}{80} = \frac{k}{10^n}$, and then the decimal form of the fraction.
  - $\frac{7}{80} = \frac{7}{2^4 \times 5} = \frac{7 \times 5^3}{2^4 \times 5 \times 5^3} = \frac{7 \times 125}{(2 \times 5)^4} = \frac{875}{10^4} = 0.0875$

Example 4 (4 minutes)

Example 4

Write $\frac{3}{160}$ as a decimal. Will it be finite or infinite? Explain.

- Let's write $\frac{3}{160}$ as a decimal. Will it be finite or infinite? Explain.
  - *We know that the fraction $\frac{3}{160}$ is equal to a finite decimal because the denominator 160 is a product of 2's and 5's. Specifically, $160 = 2^5 \times 5$.*
- What will we need to multiply $2^5 \times 5$ by so that is it equal to $(2 \times 5)^n = 10^n$?
  - *We will need to multiply by 5^4 so that $2^5 \times 5^5 = (2 \times 5)^5 = 10^5$.*
- Begin with $\frac{3}{160} = \frac{3}{2^5 \times 5}$, use what you know about equivalent fractions to rewrite $\frac{3}{160} = \frac{k}{10^n}$ and then the decimal form of the fraction.
  - $\frac{3}{160} = \frac{3}{2^5 \times 5} = \frac{3 \times 5^4}{2^5 \times 5 \times 5^4} = \frac{3 \times 625}{(2 \times 5)^5} = \frac{1875}{10^5} = 0.01875$

Exercises 11–13 (5 minutes)

Students complete Exercises 11–13 independently.

Exercises 11–13

Show your steps, but use a calculator for the multiplications.

11. Convert the fraction $\frac{37}{40}$ to a decimal.

a. Write the denominator as a product of 2's and/or 5's. Explain why this way of rewriting the denominator helps to find the decimal representation of $\frac{37}{40}$.

The denominator $40 = 2^3 \times 5$. It is helpful to know that $40 = 2^3 \times 5$ because it shows by how many factors of 5 the numerator and denominator will need to be multiplied to produce an equivalent fraction with a denominator that is a multiple of 10. When the denominator is a multiple of 10, the fraction can easily be written as a decimal using what I know about place value.

b. Find the decimal representation of $\frac{37}{40}$. Explain why your answer is reasonable.

$$\frac{37}{40} = \frac{37}{2^3 \times 5} = \frac{37 \times 5^2}{2^3 \times 5 \times 5^2} = \frac{925}{10^3} = 0.925$$

The answer is reasonable because the decimal value, 0.925, is less than one just like the fraction $\frac{37}{40}$. Also, it is reasonable and correct because the fraction $\frac{925}{1{,}000} = \frac{37}{40}$; therefore, it has the decimal expansion 0.925.

12. **Convert the fraction $\frac{3}{250}$ to a decimal.**

The denominator $250 = 2 \times 5^3$.

$$\frac{3}{250} = \frac{3}{2 \times 5^3} = \frac{3 \times 2^2}{2 \times 2^2 \times 5^3} = \frac{12}{10^3} = 0.012$$

13. **Convert the fraction $\frac{7}{1{,}250}$ to a decimal.**

The denominator $1,250 = 2 \times 5^4$.

$$\frac{7}{1,250} = \frac{7}{2 \times 5^4} = \frac{7 \times 2^3}{2 \times 2^3 \times 5^4} = \frac{56}{10^4} = 0.0056$$

Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that finite decimals are fractions with denominators that can be expressed as products of 2's and 5's.
- We know how to use equivalent fractions to convert a fraction to its decimal equivalent.
- We know that infinite decimals are those that repeat, like $0.\overline{3}$ or decimals that do not repeat, but do not terminate, such as π.

Lesson Summary

Fractions with denominators that can be expressed as products of 2's and/or 5's have decimal expansions that are finite.

Example:

Does the fraction $\frac{1}{8}$ have a finite or infinite decimal expansion?

Since $8 = 2^3$, then the fraction has a finite decimal expansion. The decimal expansion is found by:

$$\frac{1}{8} = \frac{1}{2^3} = \frac{1 \times 5^3}{2^3 \times 5^3} = \frac{125}{10^3} = 0.125$$

When the denominator of a fraction cannot be expressed as a product of 2's and/or 5's then the decimal expansion of the number will be infinite.

When infinite decimals repeat, such as $0.8888888\ldots$ or $0.454545454545\ldots$, they are typically abbreviated using the notation $0.\overline{8}$ and $0.\overline{45}$, respectively. The notation indicates that the digit 8 repeats indefinitely and that the two-digit block 45 repeats indefinitely.

Exit Ticket (4 minutes)

Name ______________________________ Date ____________________

Lesson 6: Finite and Infinite Decimals

Exit Ticket

Convert each fraction to a finite decimal. If the fraction cannot be written as a finite decimal, and then state how you know. Show your steps, but use a calculator for the multiplications.

1. $\frac{9}{16}$

2. $\frac{8}{125}$

3. $\frac{4}{15}$

4. $\frac{1}{200}$

Exit Ticket Sample Solutions

Convert each fraction to a finite decimal. If the fraction cannot be written as a finite decimal, and then state how you know. Show your steps, but use a calculator for the multiplications.

1. $\frac{9}{16}$

   The denominator $16 = 2^4$.

$$\frac{9}{16} = \frac{9}{2^4} = \frac{9 \times 5^4}{2^4 \times 5^4} = \frac{9 \times 625}{10^4} = \frac{5625}{10^4} = 0.5625$$

2. $\frac{8}{125}$

   The denominator $125 = 5^3$.

$$\frac{8}{125} = \frac{8}{5^3} = \frac{8 \times 2^3}{5^3 \times 2^3} = \frac{8 \times 8}{10^3} = \frac{64}{10^3} = 0.064$$

3. $\frac{4}{15}$

   The fraction $\frac{4}{15}$ *is not a finite decimal because the denominator* $15 = 5 \times 3$. *Since the denominator cannot be expressed as a product of* 2*'s and* 5*'s, then* $\frac{4}{15}$ *is not a finite decimal.*

4. $\frac{1}{200}$

   The denominator $200 = 2^3 \times 5^2$.

$$\frac{1}{200} = \frac{1}{2^3 \times 5^2} = \frac{1 \times 5}{2^3 \times 5^2 \times 5} = \frac{5}{2^3 \times 5^3} = \frac{5}{10^3} = 0.005$$

Problem Set Sample Solutions

Convert each fraction to a finite decimal. If the fraction cannot be written as a finite decimal, then state how you know. Show your steps, but use a calculator for the multiplications.

1. $\frac{2}{32}$

   The fraction $\frac{2}{32}$ *simplifies to* $\frac{1}{16}$.

   The denominator $16 = 2^4$.

$$\frac{1}{16} = \frac{1}{2^4} = \frac{1 \times 5^4}{2^4 \times 5^4} = \frac{625}{10^4} = 0.0625$$

2. $\frac{99}{125}$

a. Write the denominator as a product of 2's and/or 5's. Explain why this way of rewriting the denominator helps to find the decimal representation of $\frac{99}{125}$.

The denominator $125 = 5^3$. It is helpful to know that $125 = 5^3$ because it shows by how many factors of 2 the numerator and denominator will need to be multiplied to produce an equivalent fraction with a denominator that is a multiple of 10. When the denominator is a multiple of 10, the fraction can easily be written as a decimal using what I know about place value.

b. Find the decimal representation of $\frac{99}{125}$. Explain why your answer is reasonable.

$$\frac{99}{125} = \frac{99}{5^3} = \frac{99 \times 2^3}{2^3 \times 5^3} = \frac{792}{10^3} = 0.792$$

The answer is reasonable because the decimal value, 0.792, is less than one just like the fraction $\frac{99}{125}$. Also, it is reasonable and correct because the fraction $\frac{792}{1000} = \frac{99}{125}$; therefore, it has the decimal expansion 0.792.

3. $\frac{15}{128}$

The denominator $128 = 2^7$.

$$\frac{15}{128} = \frac{15}{2^7} = \frac{15 \times 5^7}{2^7 \times 5^7} = \frac{1171875}{10^7} = 0.1171875$$

4. $\frac{8}{15}$

The fraction $\frac{8}{15}$ is not a finite decimal because the denominator $15 = 3 \times 5$. Since the denominator cannot be expressed as a product of 2's and 5's, then $\frac{8}{15}$ is not a finite decimal.

5. $\frac{3}{28}$

The fraction $\frac{3}{28}$ is not a finite decimal because the denominator $28 = 2^2 \times 7$. Since the denominator cannot be expressed as a product of 2's and 5's, then $\frac{3}{28}$ is not a finite decimal.

6. $\frac{13}{400}$

The denominator $400 = 2^4 \times 5^2$.

$$\frac{13}{400} = \frac{13}{2^4 \times 5^2} = \frac{13 \times 5^2}{2^4 \times 5^2 \times 5^2} = \frac{325}{10^4} = 0.0325$$

8. $\frac{5}{64}$

The denominator $64 = 2^6$.

$$\frac{5}{64} = \frac{5}{2^6} = \frac{5 \times 5^6}{2^6 \times 5^6} = \frac{78125}{10^6} = 0.078125$$

9. $\frac{15}{35}$

The fraction $\frac{15}{35}$ *reduces to* $\frac{3}{7}$. *The denominator* 7 *cannot be expressed as a product of* 2*'s and* 5*'s. Therefore,* $\frac{3}{7}$ *is not a finite decimal.*

10. $\frac{199}{250}$

The denominator $250 = 2 \times 5^3$.

$$\frac{199}{250} = \frac{199}{2 \times 5^3} = \frac{199 \times 2^2}{2 \times 2^2 \times 5^3} = \frac{796}{10^3} = 0.796$$

11. $\frac{219}{625}$

The denominator $625 = 5^4$.

$$\frac{219}{625} = \frac{219}{5^4} = \frac{219 \times 2^4}{2^4 \times 5^4} = \frac{3504}{10^4} = 0.3504$$