



Lesson 19: The Graph of a Linear Equation in Two Variables is a Line

Student Outcomes

- Students prove that any point on the graph of $y = mx + b$ is on a line l and that any point on a line l is a point on the graph of $y = mx + b$.
- Students graph linear equations on the coordinate plane.

Lesson Notes

Included in this and the next several lessons are an analytical approach to understanding linear equations and their graphs. Though the analytical understandings are not *required* for Grade 8 students, it is of tremendous value for students to understand the proofs of common assertions about lines. For that reason, these lessons are structured to offer extensions (i.e., the proofs) in cases where the students are ready, and also to provide alternate activities that deepen conceptual understanding. The theorems are presented to students and, based on the readiness of each class, the teacher can choose to discuss the proof of the theorem or have students explore the theorem experimentally by completing the exercises of the alternate activity.

This lesson is the result of many lessons leading up to this one. In previous lessons, it was stated that there were four things that needed to be proved about lines before we could state, definitively, that the graph of a linear equation in two variables is in fact a line.

1. We must define a number for each non-vertical line that can be used to measure the “steepness” or “slant” of the line. Once defined, this number will be called the *slope* of the line and is often referred to as the “rate of change”.
2. We must show that *any* two points on a non-vertical line can be used to find the slope of the line.
3. We must show that the line joining two points on the graph of a linear equation of the form $y = mx + b$ has slope m .
4. We must show that there is only one line passing through a given point with a given slope.

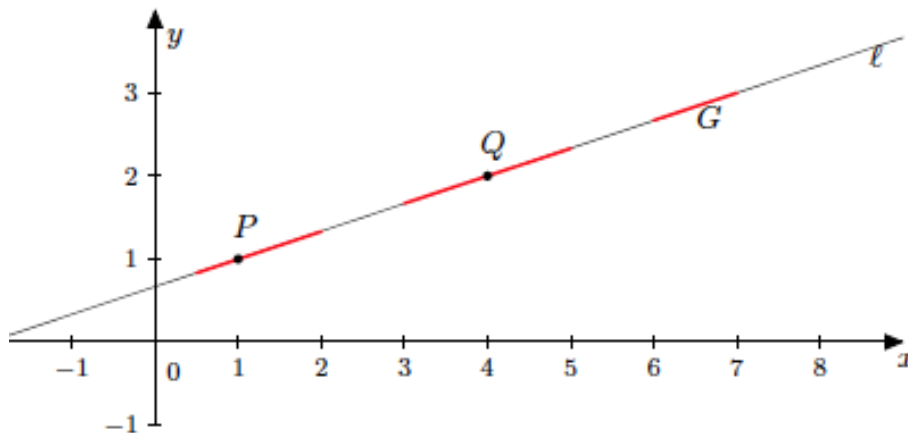
This lesson is the conclusion of that work, and it is an important result for students to observe. There are two options for the first part of the lesson. You may choose to use the formal proof in the first discussion beginning on page 281 and Exercise 1 beginning on page 283. This lesson path skips Exercises 2–5 and resumes the lesson with Example 1 on page 291. The other option is to skip the Discussion and Exercise 1 on the first five pages, having students examine several equations in the form of $ax + by = c$, create their graphs, and informally conclude that the graphs of equations of this type always result in lines. The alternate activity begins on page 285 of this lesson with Exercises 2–8. Following these exercises is Example 1 on page 291 and Exercises 9–11 where students learn how to graph a linear equation using intercepts. Students need graph paper to complete the Exercises and the Problem Set.

Classwork

Discussion (15 minutes)

Now that all of the necessary tools have been developed, we are ready to show that the graph of a linear equation in two variables in the form $y = mx + b$, where b is always a constant, is a non-vertical line.

- To prove that the graph of a linear equation is a line, we need to show the following:
 - any point on the graph of $y = mx + b$ is a point on a line l , and
 - any point on the line l is a point on the graph of $y = mx + b$.
- Why do we need to show both (1) and (2)? Consider the following: Let P and Q be two distinct points on the graph of $y = mx + b$. Let l be the line passing through points P and Q . Also, let the graph of $y = mx + b$ be the collection of red segments denoted by G , as shown.



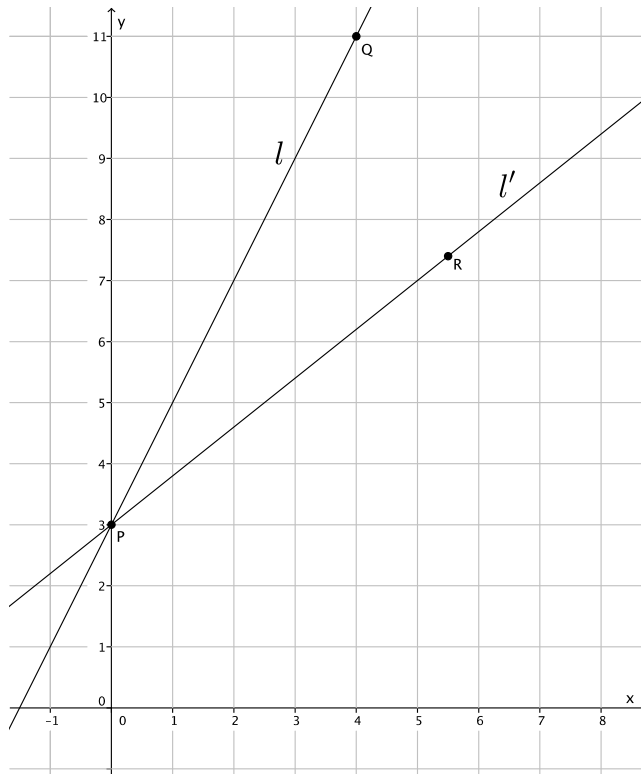
If G is the collection of red segments, as shown, then (1) is true, but (2) is not. Therefore, if we can show that both (1) and (2) are true, then we know that the graph of $y = mx + b$ is exactly the same as the graph of line l .

- Let's look at a specific case before proving (1) and (2) in general. Show that the graph of $y = 2x + 3$ is a non-vertical line.
- We need two points that we know are on the graph of $y = 2x + 3$. Are points $(0, 3)$ and $(4, 11)$ solutions to the equation $y = 2x + 3$? Explain.
 - Yes, because $3 = 2(0) + 3$ and $11 = 2(4) + 3$ are true statements.
- Let line l pass through the points $P = (0, 3)$ and $Q = (1, 5)$. By our work in Lesson 17, we know that an equation in the form $y = mx + b$ has slope m . Then, the slope of $y = 2x + 3$ is 2 and therefore the slope of line l is 2. We claim that the graph of $y = 2x + 3$ is the line l .

MP.2

MP.2

- Now we will show that *any* point on the graph of $y = 2x + 3$ is on line l . Let R be any point on the graph of $y = 2x + 3$ (other than points P and Q because if $R = P$ or if $R = Q$ there would be nothing to prove; we already showed that they are solutions to the equation $y = 2x + 3$). Let l' be the line passing through points P and R , as shown.



- We need to show that line l' has a slope of 2. We have done this before in Lesson 16. Since P and R are on the graph of $y = 2x + 3$, then the coordinates of $P = (p_1, 2p_1 + 3)$ and $R = (r_1, 2r_1 + 3)$. We also know the following:

$$\begin{aligned} \text{slope of } l' &= \frac{(2p_1 + 3) - (2r_1 + 3)}{p_1 - r_1} \\ &= \frac{2p_1 - 2r_1}{p_1 - r_1} \\ &= \frac{2(p_1 - r_1)}{p_1 - r_1} \\ &= 2 \end{aligned}$$

- Now that we know the slope of l' is 2, what can we say about lines l and l' ? Hint: What do we know about lines that have the same slope and pass through the same point?
 - Lines l and l' must be the same line. There can be only one line with a slope of 2 that passes through a given point, in this case P .*



- Therefore, point R must be on line l and the proof that any point on the graph of $y = 2x + 3$ is on line l is finished.
- Now we must show that *any* point on l is on the graph of $y = 2x + 3$. Let S be *any* point on l (except for P and Q for the same reason as before). Using $S = (s_1, s_2)$ and $P = (0, 3)$ in the slope formula, we get the following:

$$\text{slope of } l = \frac{s_2 - 3}{s_1 - 0}$$

We know the slope of l is 2.

$$\begin{aligned} 2 &= \frac{s_2 - 3}{s_1 - 0} \\ 2(s_1 - 0) &= s_2 - 3 \\ 2s_1 &= s_2 - 3 \\ 2s_1 + 3 &= s_2 - 3 + 3 \\ 2s_1 + 3 &= s_2 \end{aligned}$$

The equation above shows that point $S = (s_1, s_2)$ is a solution to the equation $y = 2x + 3$ and must be on the graph of $y = 2x + 3$.

- We have just shown that any point R on the graph of $y = 2x + 3$ is on line l and any point S on line l is on the graph of $y = 2x + 3$. Therefore, the proof is complete.

Exercise 1 (10 minutes)

Students work in pairs or small groups to prove the theorem in general. This is an optional exercise and related directly to the optional discussion above. If you chose the alternative path for this lesson, then this exercise should not be assigned.

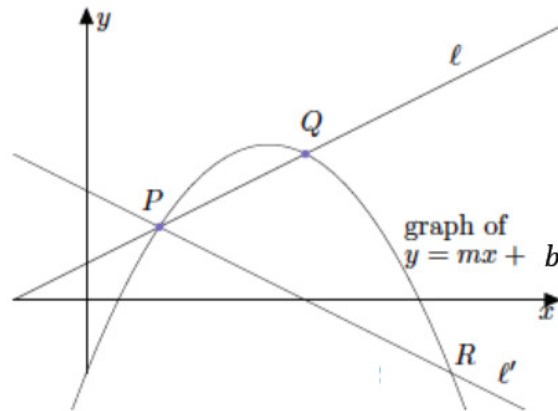
Exercise 1

Theorem: The graph of a linear equation $y = mx + b$ is a non-vertical line with slope m and passing through $(0, b)$, where b is a constant.

- Prove the theorem by completing parts (a)–(c). Given two distinct points, P and Q , on the graph of $y = mx + b$ and let l be the line passing through P and Q . You must show:
 - any point on the graph of $y = mx + b$ is on line l , and
 - any point on the line l is on the graph of $y = mx + b$.

MP.2

- a. Proof of (1): Let R be any point on the graph of $y = mx + b$. Show that R is on l . Begin by assuming it is not. Assume the graph looks like the diagram below where R is on l' .



What is the slope of line l ?

Since the points P and Q are on the graph of $y = mx + b$, then we know that the slope of the line passing through those points must have slope m . Therefore, line l has slope m .

What is the slope of line l' ?

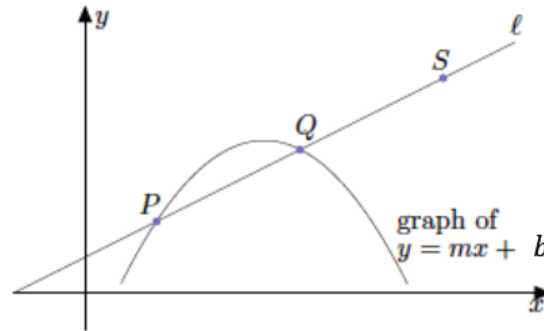
We know that point R is on the graph of $y = mx + b$. Then the coordinates of point $R = (r_1, mr_1 + b)$ because R is a solution to $y = mx + b$ and $r_2 = mr_1 + b$. Similarly, the coordinates of $P = (p_1, mp_1 + b)$. Then,

$$\begin{aligned} \text{slope of } l' &= \frac{(mp_1 + b) - (mr_1 + b)}{p_1 - r_1} \\ &= \frac{mp_1 + b - mr_1 - b}{p_1 - r_1} \\ &= \frac{mp_1 - mr_1}{p_1 - r_1} \\ &= \frac{m(p_1 - r_1)}{p_1 - r_1} \\ &= m \end{aligned}$$

What can you conclude about lines l and l' ? Explain.

Lines l and l' are the same line. Both lines go through point P and have slope m . There can be only one line with a given slope going through a given point; therefore, line $l = l'$.

b. Proof of (2): Let S be any point on line l , as shown.



Show that S is a solution to $y = mx + b$. Hint: Use the point $(0, b)$.

Point S is on line l . Let $S = (s_1, s_2)$. Then,

$$\text{slope of } l = \frac{s_2 - b}{s_1 - 0}$$

We know the slope of l is m ; therefore,

$$\begin{aligned} m &= \frac{s_2 - b}{s_1 - 0} \\ m(s_1 - 0) &= s_2 - b \\ ms_1 &= s_2 - b \\ ms_1 + b &= s_2 - b + b \\ ms_1 + b &= s_2 \\ s_2 &= ms_1 + b \end{aligned}$$

which shows S is a solution to $y = mx + b$.

c. Now that you have shown that any point on the graph of $y = mx + b$ is on line l (part a), and any point on line l is on the graph of $y = mx + b$ (part b), what can you conclude about the graphs of linear equations?

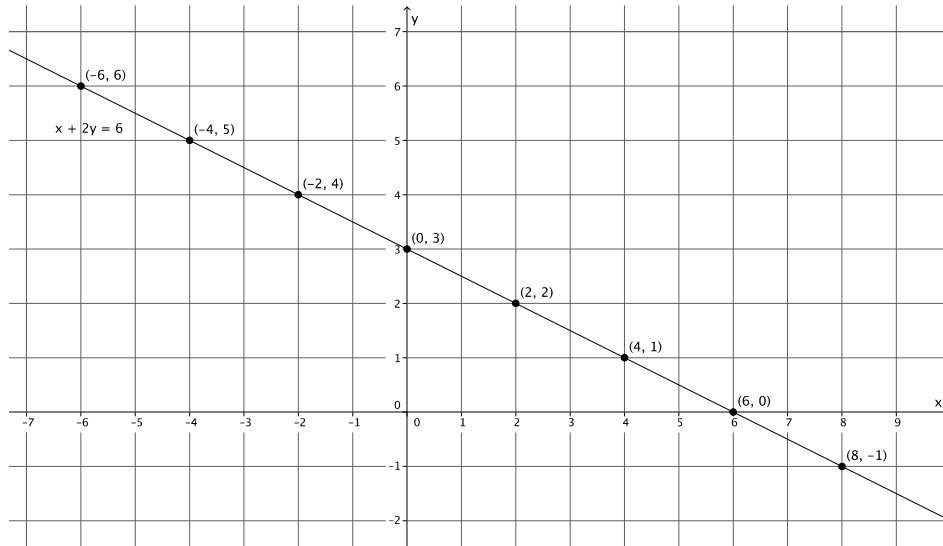
The graph of a linear equation is a line.

Exercises 2–8 (25 minutes)

These exercises should be completed in place of the Discussion and Exercise 1 above. Students need graph paper to complete the following exercises. This is the alternate activity that was described in the Lesson Notes. For these exercises, students graph a linear equation and informally verify that any point on the graph of the equation is on the line and that any point on the line is a point on the graph of the equation. Exercises 6–8 ask students to draw these conclusions based on their work in Exercises 2–5. Consider having a class discussion based on Exercises 6–8, where students share their conclusions. End the discussion by making it clear that the graph of a linear equation is a line. Then proceed with Example 1 and Exercises 9–11.

2. Use $x = 4$ and $x = -4$ to find two solutions to the equation $x + 2y = 6$. Plot the solutions as points on the coordinate plane and connect the points to make a line.

The solutions are $(4, 1)$ and $(-4, 5)$.



- a. Identify two other points on the line with integer coordinates. Verify that they are solutions to the equation $x + 2y = 6$.

Choice of points and verifications will vary. Several possibilities are noted in the graph above.

- b. When $x = 1$, what is the value of y ? Does this solution appear to be a point on the line?

$$\begin{aligned} 1 + 2y &= 6 \\ 2y &= 5 \\ y &= \frac{5}{2} \end{aligned}$$

Yes, $(1, \frac{5}{2})$ does appear to be a point on the line.

- c. When $x = -3$, what is the value of y ? Does this solution appear to be a point on the line?

$$\begin{aligned} -3 + 2y &= 6 \\ 2y &= 9 \\ y &= \frac{9}{2} \end{aligned}$$

Yes, $(-3, \frac{9}{2})$ does appear to be a point on the line.

- d. Is the point $(3, 2)$ on the line?

No, $(3, 2)$ is not a point on the line.

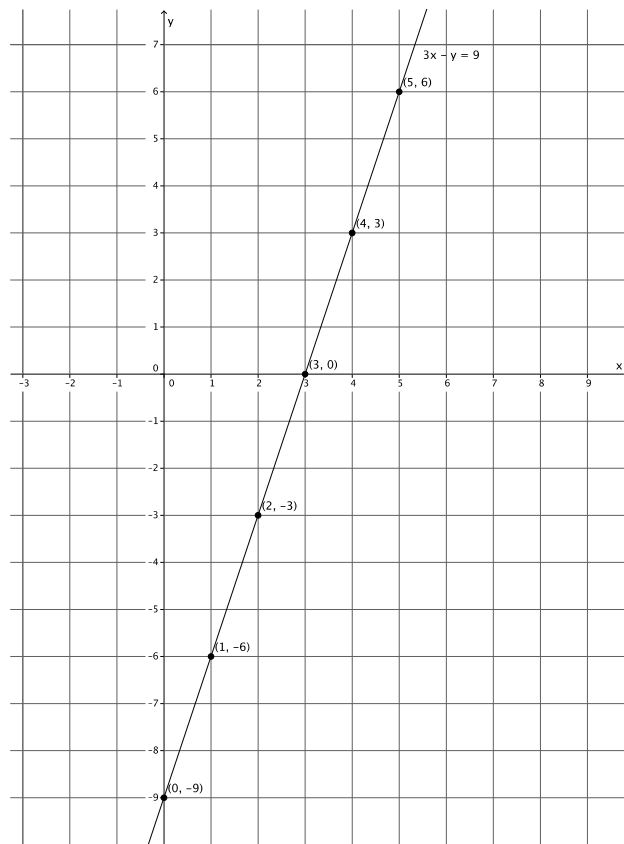
e. Is the point $(3, 2)$ a solution to the linear equation $x + 2y = 6$?

No, $(3, 2)$ is not a solution to $x + 2y = 6$ because

$$\begin{aligned} 3 + 2(2) &= 6 \\ 3 + 4 &= 6 \\ 7 &\neq 6 \end{aligned}$$

3. Use $x = 4$ and $x = 1$ to find two solutions to the equation $3x - y = 9$. Plot the solutions as points on the coordinate plane and connect the points to make a line.

The solutions are $(4, 3)$ and $(1, -6)$.



a. Identify two other points on the line with integer coordinates. Verify that they are solutions to the equation $3x - y = 9$.

Choice of points and verifications will vary. Several possibilities are noted in the graph above.

b. When $x = 4.5$, what is the value of y ? Does this solution appear to be a point on the line?

$$\begin{aligned} 3(4.5) - y &= 9 \\ 13.5 - y &= 9 \\ -y &= -4.5 \\ y &= 4.5 \end{aligned}$$

Yes, $(4.5, 4.5)$ does appear to be a point on the line.

- c. When $x = \frac{1}{2}$, what is the value of y ? Does this solution appear to be a point on the line?

$$\begin{aligned} 3\left(\frac{1}{2}\right) - y &= 9 \\ \frac{3}{2} - y &= 9 \\ -y &= 7.5 \\ y &= -7.5 \end{aligned}$$

Yes, $\left(\frac{1}{2}, -7.5\right)$ does appear to be a point on the line.

- d. Is the point $(2, 4)$ on the line?

No, $(2, 4)$ is not a point on the line.

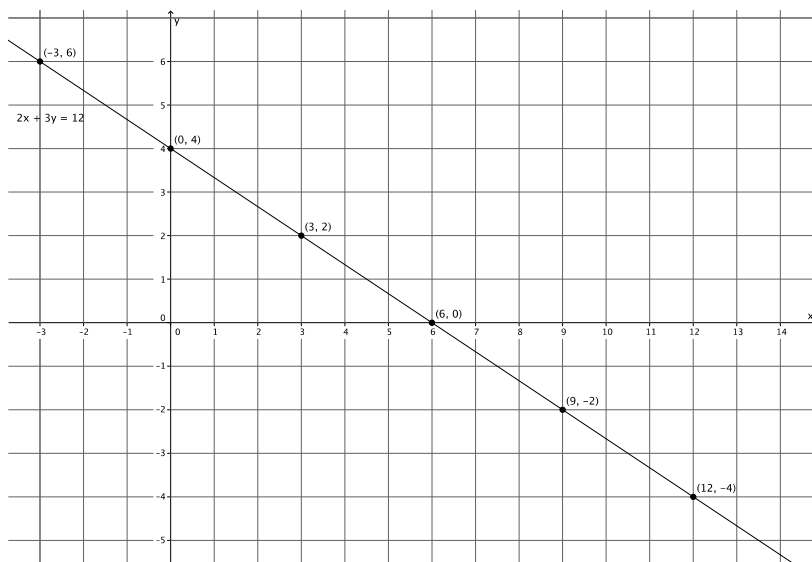
- e. Is the point $(2, 4)$ a solution to the linear equation $3x - y = 9$?

No, $(2, 4)$ is not a solution to $3x - y = 9$ because

$$\begin{aligned} 3(2) - 4 &= 9 \\ 6 - 4 &= 9 \\ 2 &\neq 9 \end{aligned}$$

4. Use $x = 3$ and $x = -3$ to find two solutions to the equation $2x + 3y = 12$. Plot the solutions as points on the coordinate plane and connect the points to make a line.

The solutions are $(-3, 6)$ and $(3, 2)$.



- a. Identify two other points on the line with integer coordinates. Verify that they are solutions to the equation $2x + 3y = 12$.

Choice of points and verifications will vary. Several possibilities are noted in the graph above.

- b. When $x = 2$, what is the value of y ? Does this solution appear to be a point on the line?

$$\begin{aligned} 2(2) + 3y &= 12 \\ 4 + 3y &= 12 \\ 3y &= 8 \\ y &= \frac{8}{3} \end{aligned}$$

Yes, $(2, \frac{8}{3})$ does appear to be a point on the line.

- c. When $x = -2$, what is the value of y ? Does this solution appear to be a point on the line?

$$\begin{aligned} 2(-2) + 3y &= 12 \\ -4 + 3y &= 12 \\ 3y &= 16 \\ y &= \frac{16}{3} \end{aligned}$$

Yes, $(-2, \frac{16}{3})$ does appear to be a point on the line.

- d. Is the point $(8, -3)$ on the line?

No, $(8, -3)$ is not a point on the line.

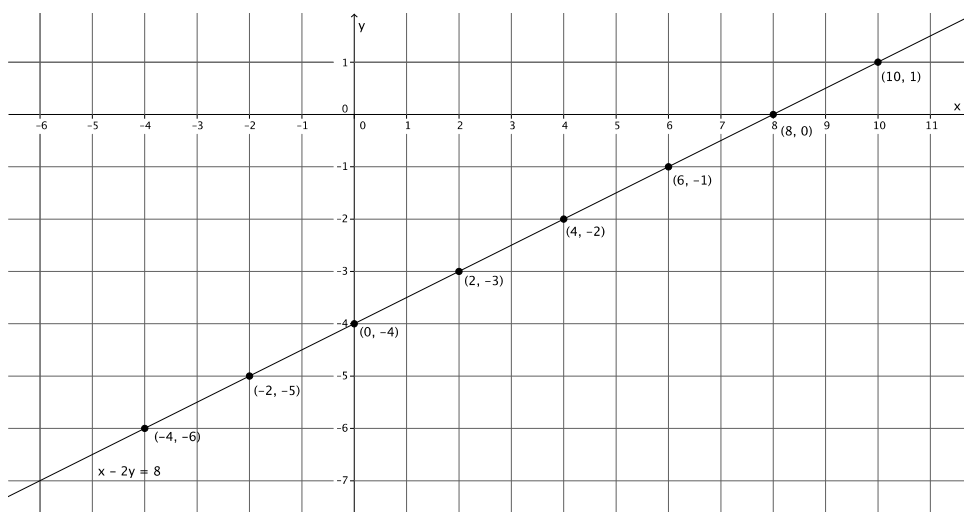
- e. Is the point $(8, -3)$ a solution to the linear equation $2x + 3y = 12$?

No, $(8, -3)$ is not a solution to $2x + 3y = 12$ because

$$\begin{aligned} 2(8) + 3(-3) &= 12 \\ 16 - 9 &= 12 \\ 7 &\neq 12 \end{aligned}$$

5. Use $x = 4$ and $x = -4$ to find two solutions to the equation $x - 2y = 8$. Plot the solutions as points on the coordinate plane and connect the points to make a line.

The solutions are $(4, -2)$ and $(-4, -6)$.



- a. Identify two other points on the line with integer coordinates. Verify that they are solutions to the equation $x - 2y = 8$.

Choice of points and verifications will vary. Several possibilities are noted in the graph above.

- b. When $x = 7$, what is the value of y ? Does this solution appear to be a point on the line?

$$\begin{aligned} 7 - 2y &= 8 \\ -2y &= 1 \\ y &= -\frac{1}{2} \end{aligned}$$

Yes, $(7, -\frac{1}{2})$ does appear to be a point on the line.

- c. When $x = -3$, what is the value of y ? Does this solution appear to be a point on the line?

$$\begin{aligned} -3 - 2y &= 8 \\ -2y &= 11 \\ y &= -\frac{11}{2} \end{aligned}$$

Yes, $(-3, -\frac{11}{2})$ does appear to be a point on the line.

- d. Is the point $(-2, -3)$ on the line?

No, $(-2, -3)$ is not a point on the line.

- e. Is the point $(-2, -3)$ a solution to the linear equation $x - 2y = 8$?

No, $(-2, -3)$ is not a solution to $x - 2y = 8$ because

$$\begin{aligned} -2 - 2(-3) &= 8 \\ -2 + 6 &= 8 \\ 4 &\neq 8 \end{aligned}$$

6. Based on your work in Exercises 2–5, what conclusions can you draw about the points on a line and solutions to a linear equation?

It appears that all points on the line represent a solution to the equation. That is, any point identified on the line is a solution to the linear equation.

7. Based on your work in Exercises 2–5, will a point that is not a solution to a linear equation be a point on the graph of a linear equation? Explain.

No, each time we were given a point off the line in part (d), we verified that it was not a solution to the equation in part (e). For that reason, I would expect that all points not on the line would not be a solution to the equation.

8. Based on your work in Exercises 2–5, what conclusions can you draw about the graph of a linear equation?

The graph of a linear equation is a line.

MP.8

Example 1 (4 minutes)

- Now that we know that the graph of a linear equation in two variables is a line and that there is only one line that can pass through two points, then we can easily graph equations using intercepts.
- We already know that the y -intercept is the location on the graph of a line where the line intersects the y -axis. The point of intersection will have coordinates $(0, y)$. Similarly, the x -intercept is the location on the graph of a line where the line intersects the x -axis. The point of intersection will have coordinates $(x, 0)$.
- To graph using intercepts, simply replace the symbols x and y with zero, one at a time, and solve.
- Graph the equation: $2x + 3y = 9$.

Replace x with zero and solve for y to determine the y -intercept:

$$\begin{aligned}2(0) + 3y &= 9 \\3y &= 9 \\y &= 3\end{aligned}$$

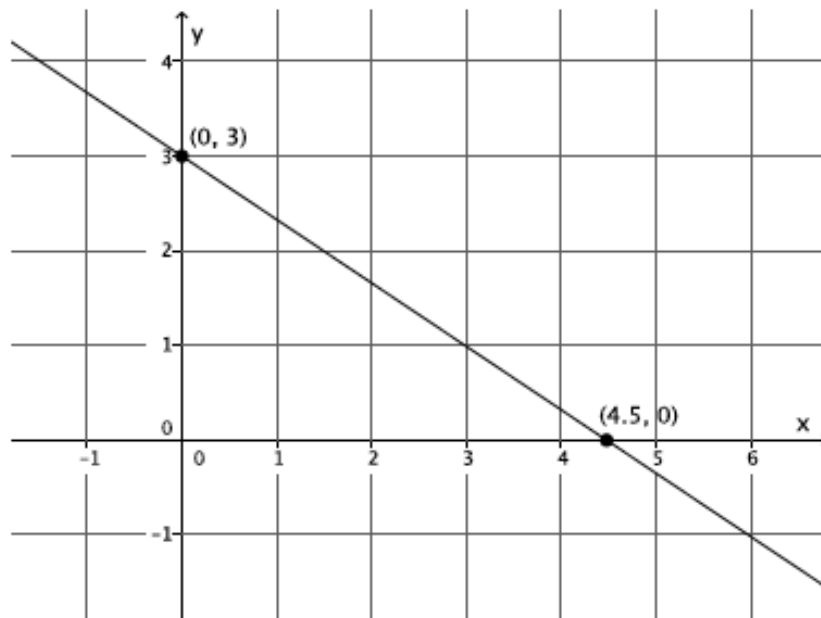
The y -intercept is at $(0, 3)$.

Replace y with zero and solve for x to determine the x -intercept:

$$\begin{aligned}2x + 3(0) &= 9 \\2x &= 9 \\x &= \frac{9}{2}\end{aligned}$$

The x -intercept is at $(\frac{9}{2}, 0)$.

- Now that we know the intercepts, we can place those two points on the graph and connect them to graph the linear equation $2x + 3y = 9$:



- Graphing using intercepts is an efficient way of graphing linear equations that are in standard form. Graphing using the slope and y -intercept is the most efficient way of graphing linear equations that are in slope-intercept form. Creating a table and finding solutions is another way we have learned how to graph linear equations. All three methods work, but some methods will save time depending on the form of the equation.

Exercise 9–11 (5 minutes)

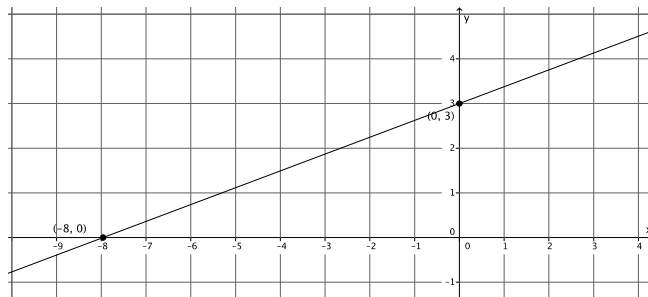
9. Graph the equation $-3x + 8y = 24$ using intercepts.

$$\begin{aligned} -3x + 8y &= 24 \\ -3(0) + 8y &= 24 \\ 8y &= 24 \\ y &= 3 \end{aligned}$$

The y -intercept is $(0, 3)$.

$$\begin{aligned} -3x + 8y &= 24 \\ -3x + 8(0) &= 24 \\ -3x &= 24 \\ x &= -8 \end{aligned}$$

The x -intercept is $(-8, 0)$.



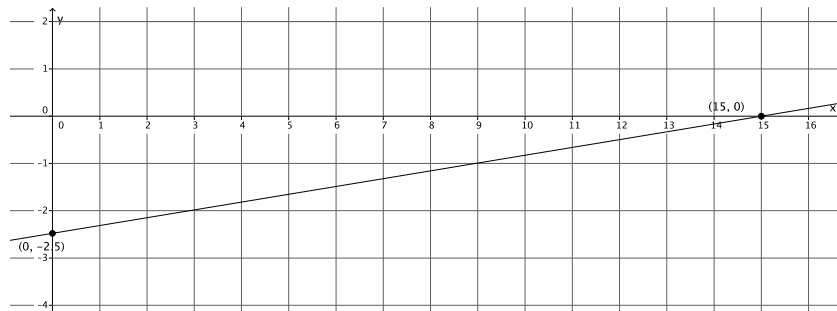
10. Graph the equation $x - 6y = 15$ using intercepts.

$$\begin{aligned} x - 6y &= 15 \\ 0 - 6y &= 15 \\ -6y &= 15 \\ y &= -\frac{15}{6} \\ y &= -\frac{5}{2} \end{aligned}$$

The y -intercept is $(0, -\frac{5}{2})$.

$$\begin{aligned} x - 6y &= 15 \\ x - 6(0) &= 15 \\ x &= 15 \end{aligned}$$

The x -intercept is $(15, 0)$.



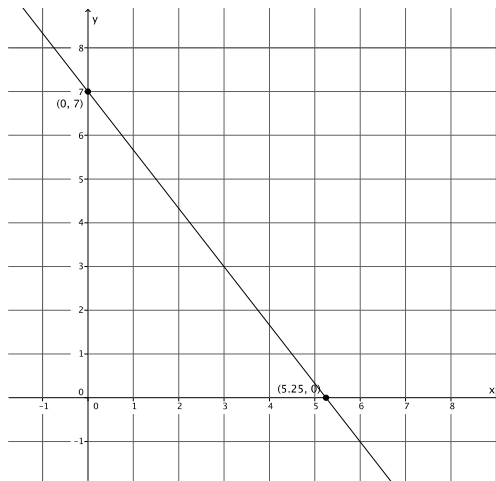
11. Graph the equation $4x + 3y = 21$ using intercepts.

$$\begin{aligned}4x + 3y &= 21 \\4(0) + 3y &= 21 \\3y &= 21 \\y &= 7\end{aligned}$$

The *y*-intercept is $(0, 7)$.

$$\begin{aligned}4x + 3y &= 21 \\4x + 3(0) &= 21 \\4x &= 21 \\x &= \frac{21}{4}\end{aligned}$$

The *x*-intercept is $(\frac{21}{4}, 0)$.



Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that to prove the graph of a linear equation is a line we have to show that any point on the graph of the equation is on the line, and any point on the line is on the graph of the equation.
- We can use what we know about slope and the fact that there is only one line with a given slope that goes through a given point, to prove that the graph of a linear equation is a line.
- We have another method for graphing linear equations: using intercepts.

Lesson Summary

The graph of a linear equation is a line. A linear equation can be graphed using two-points: the x -intercept and the y -intercept.

Example:

Graph the equation: $2x + 3y = 9$.

Replace x with zero and solve for y to determine the y -intercept:

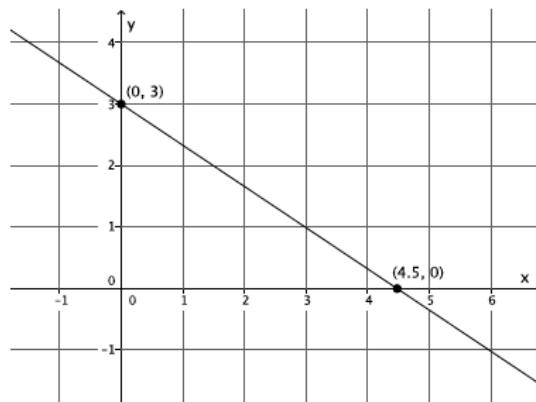
$$\begin{aligned}2(0) + 3y &= 9 \\3y &= 9 \\y &= 3\end{aligned}$$

The y -intercept is at $(0, 3)$.

Replace y with zero and solve for x to determine the x -intercept:

$$\begin{aligned}2x + 3(0) &= 9 \\2x &= 9 \\x &= \frac{9}{2}\end{aligned}$$

The x -intercept is at $(\frac{9}{2}, 0)$.

**Exit Ticket (5 minutes)**



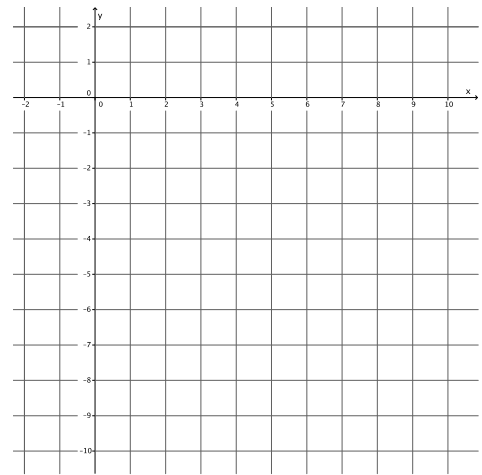
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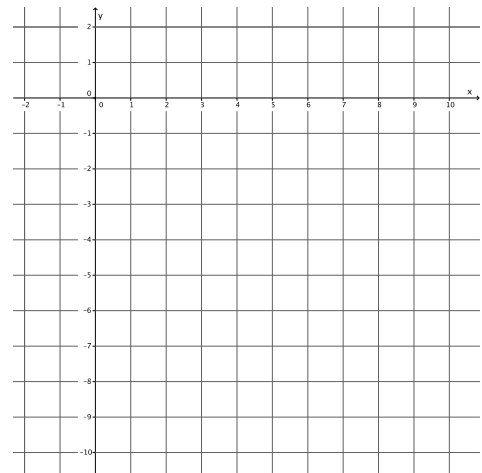
Lesson 19: The Graph of a Linear Equation in Two Variables is a Line

Exit Ticket

1. Graph the equation $y = \frac{5}{4}x - 10$ using the y -intercept and slope.



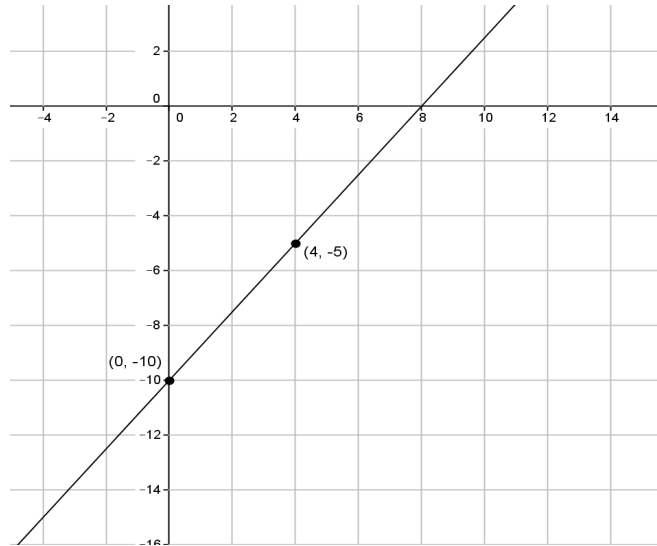
2. Graph the equation $5x - 4y = 40$ using intercepts.



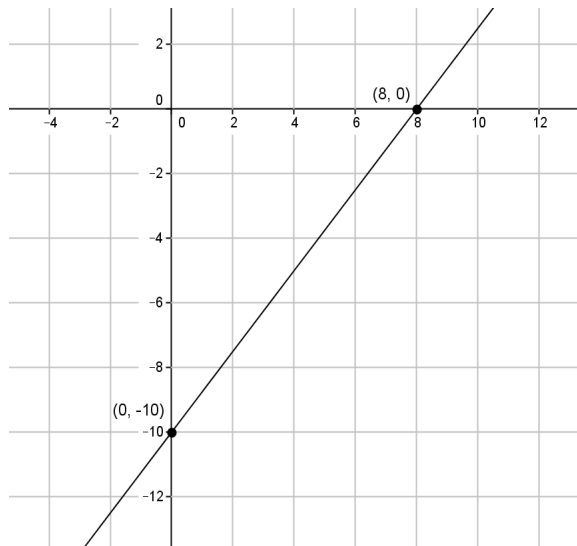
3. What can you conclude about the equations $y = \frac{5}{4}x - 10$ and $5x - 4y = 40$?

Exit Ticket Sample Solutions

1. Graph the equation $y = \frac{5}{4}x - 10$ using the y-intercept and slope.



2. Graph the equation $5x - 4y = 40$ using intercepts.



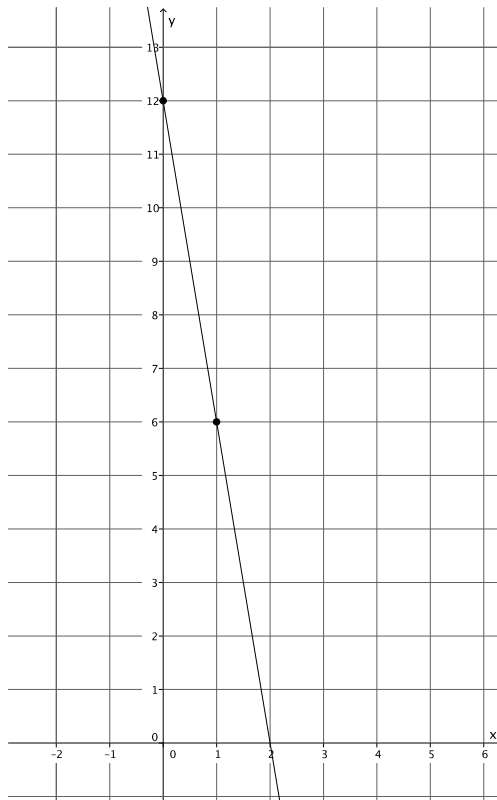
3. What can you conclude about the equations $y = \frac{5}{4}x - 10$ and $5x - 4y = 40$?

Since the points $(0, -10)$, $(4, -5)$, and $(8, 0)$ are common to both graphs, then the lines must be the same. There is only one line that can pass through two points. If you transform the equation $y = \frac{5}{4}x - 10$ so that it is in standard form, it is the equation $5x - 4y = 40$.

Problem Set Sample Solutions

Now that students know the graph of a linear equation is a line, students practice graphing linear equations in two variables using an appropriate method. Students need graph paper to complete the Problem Set.

1. Graph the equation: $y = -6x + 12$.



2. Graph the equation: $9x + 3y = 18$.

$$9(0) + 3y = 18$$

$$3y = 18$$

$$y = 6$$

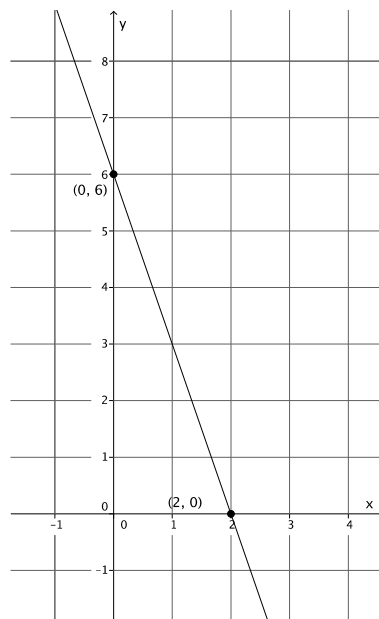
The *y*-intercept is (0, 6).

$$9x + 3(0) = 18$$

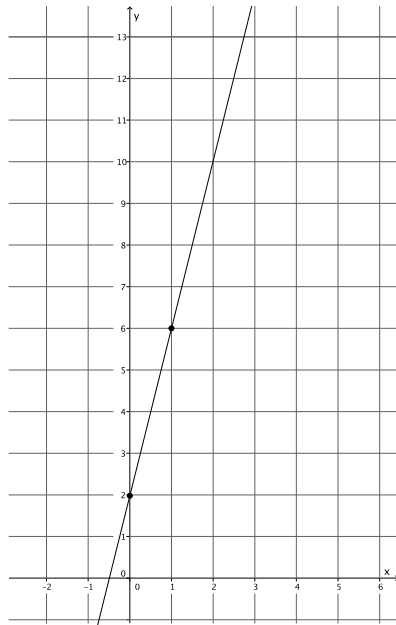
$$9x = 18$$

$$x = 2$$

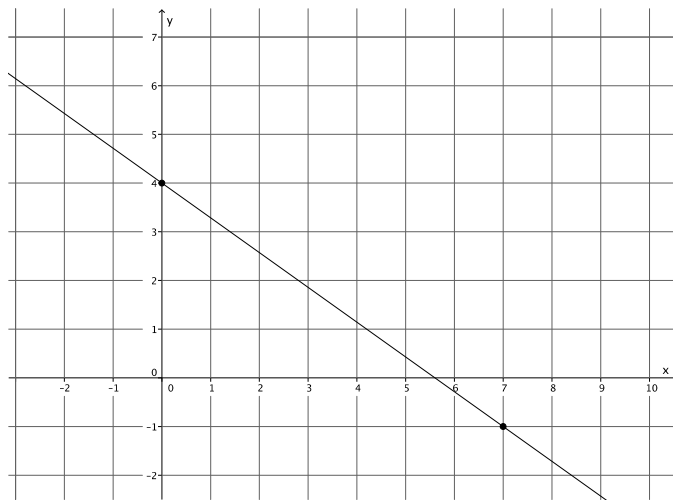
The *x*-intercept is (2, 0).



3. Graph the equation: $y = 4x + 2$.



4. Graph the equation: $y = -\frac{5}{7}x + 4$.



5. Graph the equation:

$$\frac{3}{4}x + y = 8.$$

$$\frac{3}{4}(0) + y = 8$$

$$y = 8$$

The *y*-intercept is (0, 8).

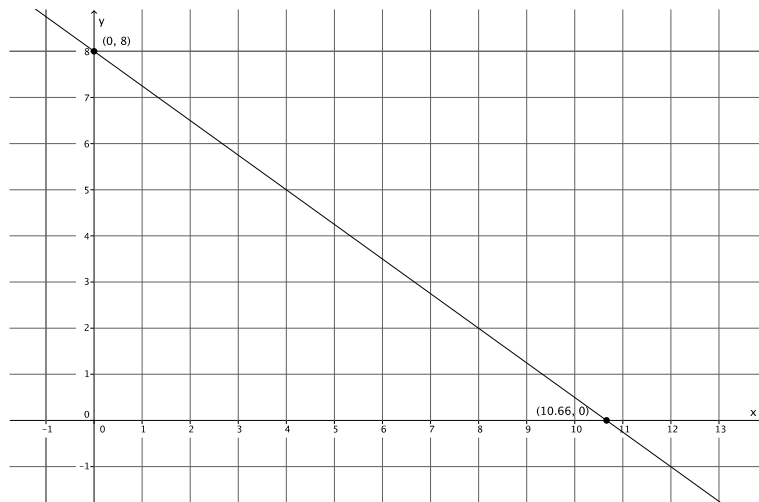
$$\frac{3}{4}x + 0 = 8$$

$$\frac{3}{4}x = 8$$

$$x = \frac{32}{3}$$

The *x*-intercept

is $(\frac{32}{3}, 0)$.



6. Graph the equation: $2x - 4y = 12$.

$$2(0) - 4y = 12$$

$$-4y = 12$$

$$y = -3$$

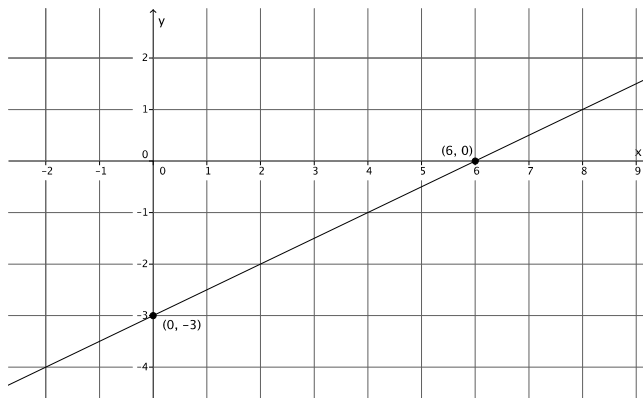
The *y*-intercept is (0, -3).

$$2x - 4(0) = 12$$

$$2x = 12$$

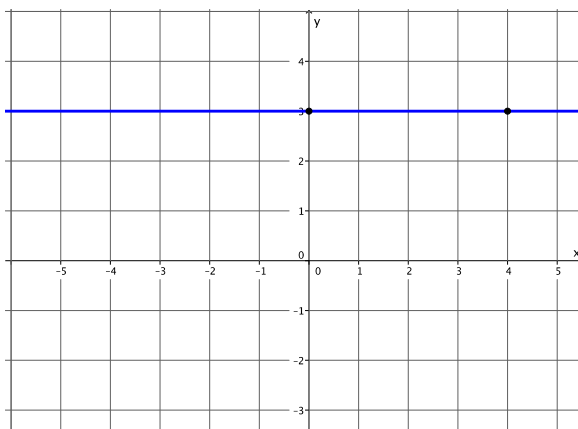
$$x = 6$$

The *x*-intercept is (6, 0).



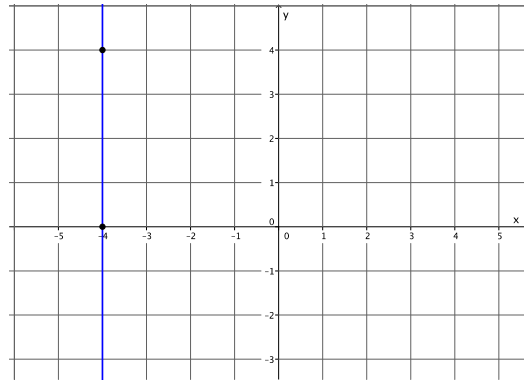
7. Graph the equation: $y = 3$. What is the slope of the graph of this line?

The slope of this line is zero.



8. Graph the equation: $x = -4$. What is the slope of the graph of this line?

The slope of this line is undefined.



9. Is the graph of $4x + 5y = \frac{3}{7}$ a line? Explain.

Yes, the graph of $4x + 5y = \frac{3}{7}$ is a line because it is a linear equation comprised of linear expressions on both side of the equal sign.

10. Is the graph of $6x^2 - 2y = 7$ a line? Explain.

Maybe. The equation $6x^2 - 2y = 7$ is not a linear equation because the expression on the left side of the equal sign is not a linear expression. If this were a linear equation, then I would be sure that it graphs as a line, but because it is not, I'm not sure what the graph of this equation would look like.