Lesson 14: The Converse of the Pythagorean Theorem

Student Outcomes

- Students illuminate the converse of the Pythagorean Theorem through computation of examples and counterexamples.
- Students apply the theorem and its converse to solve problems.

Lesson Notes

Since 8.G.6 and 8.G.7 are post-test standards, this lesson is designated as an extension lesson for this module. However, the content within this lesson is prerequisite knowledge for Module 7. If this lesson is not used with students as part of the work within Module 3, it must be used with students prior to beginning work on Module 7. Please realize that many mathematicians agree that the Pythagorean Theorem is the most important theorem in geometry and has immense implications in much of high school mathematics in general (e.g., learning of quadratics, trigonometry, etc.). It is crucial that students see the teacher explain several proofs of the Pythagorean Theorem and practice using it before being expected to produce a proof on their own.

Classwork

Concept Development (8 minutes)

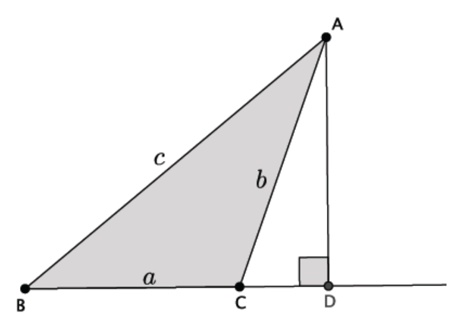
- So far, you have seen two different proofs of the Pythagorean Theorem:

  If the lengths of the legs of a right triangle are a and b, and the length of the hypotenuse is c, then $a^2 + b^2 = c^2$.
- This theorem has a converse:

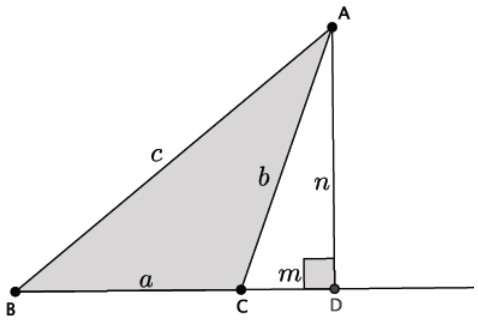
  If the lengths of three sides of a triangle, a, b, and c, satisfy $c^2 = a^2 + b^2$, then the triangle is a right triangle, and furthermore, the side of length c is opposite the right angle.

Consider an activity in which students attempt to draw a triangle on graph paper that satisfies $c^2 = a^2 + b^2$ but is not a right triangle. Students should have access to rulers for this. Activities of this type may be sufficient to develop conceptual understanding of the converse; a formal proof by contradiction follows that may also be used.

- The following is a proof of the converse. Assume we are given a triangle ABC with sides a, b, and c. We want to show that $\angle ACB$ is a right angle. To do so, we will assume that $\angle ACB$ is not a right angle. Then $|\angle ACB| > 90^\circ$ or $|\angle ACB| < 90^\circ$. For brevity, we will only show the case for when $|\angle ACB| > 90^\circ$ (the proof of the other case is similar). In the diagram below, we extend BC to a ray BC and let the perpendicular from A meet the ray at point D.



- Let $m = |CD|$ and $n = |AD|$.

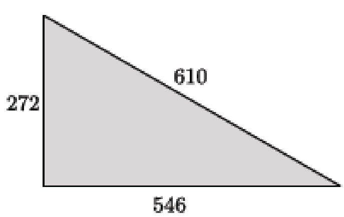


- Then by the Pythagorean Theorem applied to $\triangle ACD$ and $\triangle ABD$ results in
$$b^2 = m^2 + n^2 \text{ and } c^2 = (a+m)^2 + n^2.$$
Since we know what b^2 and c^2 are from the above equations, we can substitute those values into $c^2 = a^2 + b^2$ to get
$$(a+m)^2 + n^2 = a^2 + m^2 + n^2.$$
Since $(a+m)^2 = (a+m)(a+m) = a^2 + am + am + m^2 = a^2 + 2am + m^2$, then we have
$$a^2 + 2am + m^2 + n^2 = a^2 + m^2 + n^2.$$
We can subtract the terms a^2, m^2, and n^2 from both sides of the equal sign. Then we have
$$2am = 0.$$
- But this cannot be true because $2am$ is a length; therefore, it cannot be equal to zero. Which means our assumption that $|\angle ACB| > 90^\circ$ cannot be true. We can write a similar proof to show that $|\angle ACB| < 90^\circ$ cannot be true either. Therefore, $|\angle ACB| = 90^\circ$.

Example 1 (7 minutes)

To maintain the focus of the lesson, allow the use of calculators in order to check the validity of the right angle using the Pythagorean Theorem.

- The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle?



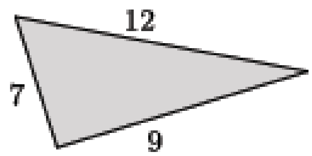
Scaffolding:
You may need to demonstrate how to use the squared button on a calculator.

- In order to find out, we need to put these numbers into the Pythagorean Theorem. Recall that side c is always the longest side. Since 610 is the largest number, it is representing the c in the Pythagorean Theorem. To determine if this triangle is a right triangle, then we need to verify the computation:
$$272^2 + 546^2 \stackrel{?}{=} 610^2$$

- Find the value of the left side of the equation: $272^2 + 546^2 = 372{,}100$. Then, find the value of the right side of the equation: $610^2 = 372{,}100$. Since the left side of the equation is equal to the right side of the equation, then we have a true statement, i.e., $272^2 + 546^2 = 610^2$. What does that mean about the triangle?
  - *It means that the triangle with side lengths of $272, 546$, and 610 is a right triangle.*

Example 2 (5 minutes)

- The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle?



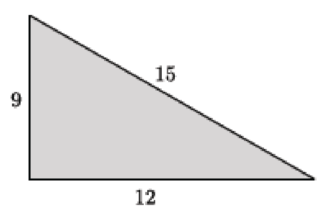
- What do we need to do to find out if this is a right triangle?
  - *We need to see if it makes a true statement when we replace a, b, and c with the numbers using the Pythagorean Theorem.*
- Which number is c? How do you know?
  - *The longest side is 12; therefore, $c = 12$.*
- Use your calculator to see if it makes a true statement. (Give students a minute to calculate.) Is it a right triangle? Explain.
  - *No, it is not a right triangle. If it were a right triangle the equation $7^2 + 9^2 = 12^2$ would be true. But the left side of the equation is equal to 130, and the right side of the equation is equal to 144. Since $130 \neq 144$, then these lengths do not form a right triangle.*

Exercises 1–7 (15 minutes)

Students complete Exercises 1–4 independently. Use of calculators is recommended.

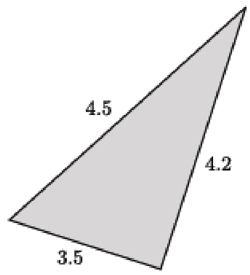
Exercises

1. The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.



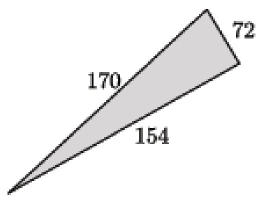
We need to check if $9^2 + 12^2 = 15^2$ is a true statement. The left side of the equation is equal to 225. The right side of the equation is equal to 225. That means $9^2 + 12^2 = 15^2$ is true, and the triangle shown is a right triangle.

2. The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.



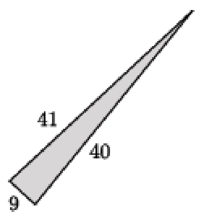
We need to check if $3.5^2 + 4.2^2 = 4.5^2$ is a true statement. The left side of the equation is equal to 29.89. The right side of the equation is equal to 20.25. That means $3.5^2 + 4.2^2 = 4.5^2$ is not true, and the triangle shown is not a right triangle.

3. The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.



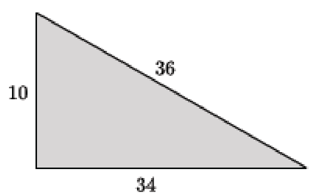
We need to check if $72^2 + 154^2 = 170^2$ is a true statement. The left side of the equation is equal to $28,900$. The right side of the equation is equal to $28,900$. That means $72^2 + 154^2 = 170^2$ is true, and the triangle shown is a right triangle.

4. The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.



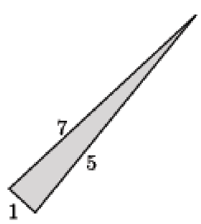
We need to check if $9^2 + 40^2 = 41^2$ is a true statement. The left side of the equation is equal to $1,681$. The right side of the equation is equal to $1,681$. That means $9^2 + 40^2 = 41^2$ is true, and the triangle shown is a right triangle.

5. The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.



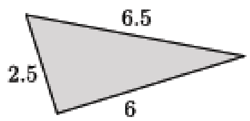
We need to check if $10^2 + 34^2 = 36^2$ is a true statement. The left side of the equation is equal to $1,256$. The right side of the equation is equal to $1,296$. That means $10^2 + 34^2 = 36^2$ is not true, and the triangle shown is not a right triangle.

6. The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.



We need to check if $1^2 + 5^2 = 7^2$ is a true statement. The left side of the equation is equal to 26. The right side of the equation is equal to 49. That means $1^2 + 5^2 = 7^2$ is not true, and the triangle shown is not a right triangle.

7. **The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.**



We need to check if $2.5^2 + 6^2 = 6.5^2$ is a true statement. The left side of the equation is equal to 42.25. The right side of the equation is equal to 42.25. That means $2.5^2 + 6^2 = 6.5^2$ is true, and the triangle shown is a right triangle.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know the converse of the Pythagorean Theorem states that if side lengths of a triangle a, b, c, satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle.
- We know that if the side lengths of a triangle a, b, c, do not satisfy $a^2 + b^2 = c^2$, then the triangle is not a right triangle.

Lesson Summary

The converse of the Pythagorean Theorem states that if side lengths of a triangle a, b, c, satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle.

If the side lengths of a triangle a, b, c, do not satisfy $a^2 + b^2 = c^2$, then the triangle is not a right triangle.

Exit Ticket (5 minutes)

Name ____ Date____________________

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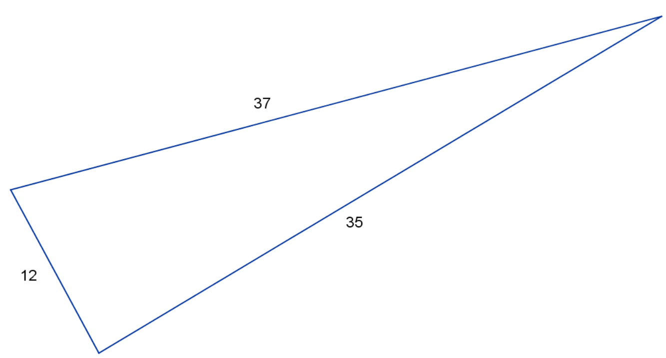
Exit Ticket

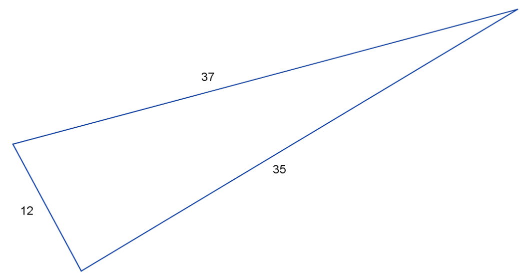
1. The numbers in the diagram below indicate the lengths of the sides of the triangle. Bernadette drew the following triangle and claims it a right triangle. How can she be sure?



2. Will the lengths 5, 9, and 14 form a right triangle? Explain.

Exit Ticket Sample Solutions

1. The numbers in the diagram below indicate the units of length of each side of the triangle. Bernadette drew the following triangle and claims it a right triangle. How can she be sure?



Since 37 is the longest side, if this triangle was a right triangle 37 would have to be the hypotenuse (or c). Now she needs to check to see if $12^2 + 35^2 = 37^2$ is a true statement. The left side is $1,369$, and the right side is $1,369$. That means $12^2 + 35^2 = 37^2$ is true, and this is a right triangle.

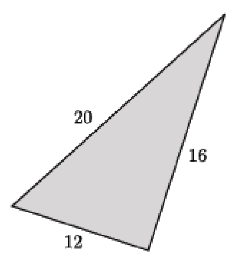
2. Will the lengths $5, 9$, and 14 form a right triangle? Explain.

No, lengths of $5, 9$, and 14 will not form a right triangle. If they did, then the equation $5^2 + 9^2 = 14^2$ would be a true statement. However, the left side equals 106, and the right side equals 196. Therefore, these lengths do not form a right triangle.

Problem Set Sample Solutions

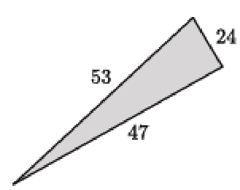
Students practice using the converse of the Pythagorean Theorem and identifying common errors in computations.

1. The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.



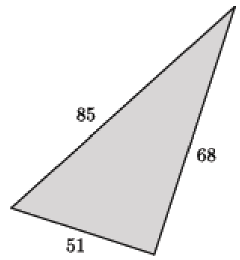
We need to check if $12^2 + 16^2 = 20^2$ is a true statement. The left side of the equation is equal to 400. The right side of the equation is equal to 400. That means $12^2 + 16^2 = 20^2$ is true, and the triangle shown is a right triangle.

2. The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.



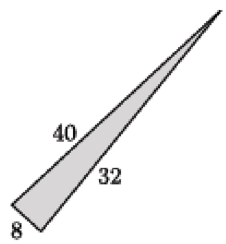
We need to check if $47^2 + 24^2 = 53^2$ is a true statement. The left side of the equation is equal to $2,785$. The right side of the equation is equal to $2,809$. That means $47^2 + 24^2 = 53^2$ is not true, and the triangle shown is not a right triangle.

3. The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.



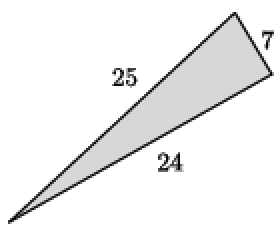
We need to check if $51^2 + 68^2 = 85^2$ is a true statement. The left side of the equation is equal to $7,225$. The right side of the equation is equal to $7,225$. That means $51^2 + 68^2 = 85^2$ is true, and the triangle shown is a right triangle.

4. The numbers in the diagram below indicate the units of length of each side of the triangle. Sam said that the following triangle is a right triangle. Explain to Sam what he did wrong to reach this conclusion and what the correct solution is.



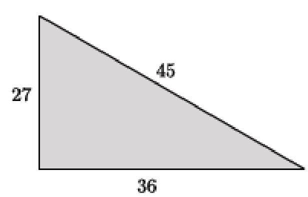
Sam forgot to square each of the side lengths. In other words, he said $8 + 32 = 40$, which is a true statement. However, to show that a triangle is a right triangle, you have to use the Pythagorean Theorem, which is $a^2 + b^2 = c^2$. Using the Pythagorean Theorem, the left side of the equation is equal to $1,088$, and the right side is equal to $1,600$. Since $1,088 \neq 1,600$, then the triangle is not a right triangle.

5. The numbers in the diagram below indicate the units of length of each side of the triangle. Is the triangle shown below a right triangle? Show your work, and answer in a complete sentence.



We need to check if $24^2 + 7^2 = 25^2$ is a true statement. The left side of the equation is equal to 625. The right side of the equation is equal to 625. That means $24^2 + 7^2 = 25^2$ is true, and the triangle shown is a right triangle.

6. Jocelyn said that the triangle below is not a right triangle. Her work is shown below. Explain what she did wrong, and show Jocelyn the correct solution.



We need to check if $27^2 + 45^2 = 36^2$ is a true statement. The left side of the equation is equal to $2,754$. The right side of the equation is equal to $1,296$. That means $27^2 + 45^2 = 36^2$ is not true, and the triangle shown is not a right triangle.

Jocelyn made the mistake of not putting the longest side of the triangle in place of c in the Pythagorean Theorem, $a^2 + b^2 = c^2$. Specifically, she should have used 45 for c, but instead she used 36 for c. If she had done that part correctly, she would have seen that, in fact, $27^2 + 36^2 = 45^2$ is a true statement because both sides of the equation equal $2,025$. That means that the triangle is a right triangle.