Lesson 3: Translating Lines

Student Outcomes

- Students learn that when lines are translated they are either parallel to the given line, or the lines coincide.
- Students learn that translations map parallel lines to parallel lines.

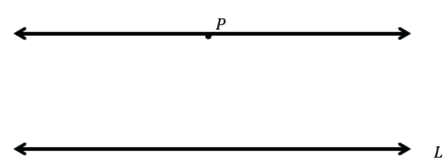
Classwork

Exercise 1 (3 minutes)

Students complete Exercise 1 independently in preparation for the Socratic discussion that follows.

Exercises

1. **Draw a line passing through point P that is parallel to line L. Draw a second line passing through point P that is parallel to line L, that is distinct (i.e., different) from the first one. What do you notice?**



Students should realize that they can only draw one line through point P that is parallel to L.

Discussion (3 minutes)

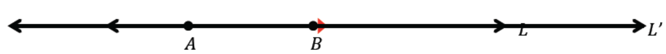
Bring out a fundamental assumption about the plane (as observed in Exercise 1):

- Given a line L and a point P not lying on L, there is at most one line passing through P and parallel to L.
  - Based on what we have learned up to now, we can cannot prove or explain this, so we have to simply agree that this is one of the starting points in the study of the plane.
  - This idea plays a key role in everything we do in the plane. A first consequence is that given a line L and a point P not lying on L, we can now refer to the line (because we agree there is only one) passing through P and parallel to L.

Exercises 2–4 (9 minutes)

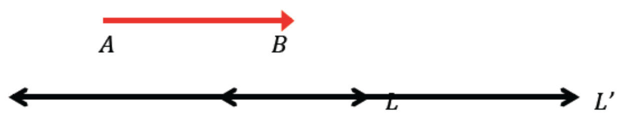
Students complete Exercises 2–4 independently in preparation for the Socratic discussion that follows.

2. Translate line L along the vector $\overrightarrow{AB}$. What do you notice about L and its image L'?



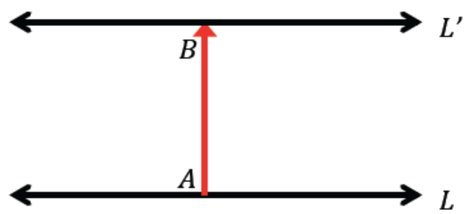
L and L' coincide, $L = L'$.

3. Line L is parallel to vector $\overrightarrow{AB}$. Translate line L along vector $\overrightarrow{AB}$. What do you notice about L and its image, L'?



L and L' coincide, again. $L = L'$.

4. Translate line L along the vector $\overrightarrow{AB}$. What do you notice about L and its image, L'?



$L \parallel L'$.

> *Scaffolding:*
> Refer to Exercises 2–4 throughout the discussion and in the summary of findings about translating lines.

Discussion (15 minutes)

- Now we examine the effect of a translation on a *line*. Thus, let line L be given. Again, let the translation be along a given $\overrightarrow{AB}$ and let L' denote the image line of the translated L. We want to know what L' is relative to AB and line L.
- If $L = L_{AB}$ or $L \parallel L_{AB}$, then $L' = L$.
  - If $L = L_{AB}$, then this conclusion follows directly from part (A) above, which says if C is on L_{AB}, then so is C' and therefore $L' = L_{AB,}$ and therefore $L = L'$ (Exercise 4).
  - If $L \parallel L_{AB}$ and C is on L, part (B) above says C' lies on the line l passing through C and parallel to L_{AB}. But L is given as a line passing through C

> *Note to Teacher:*
> We use the notation $Translation(L)$ as a precursor to the notation students will encounter in Grade 10, i.e., $T(L)$. We want to make clear the basic rigid motion that is being performed, so the notation: $Translation(L)$ is written to mean "the translation of L along the specified vector."

and parallel to L_{AB}, so the basic assumption that there is just one line passing through a point, parallel to a line (Exercise 1), implies $l = L$. Therefore, C' lies on L after all, and the translation maps every point of L to a point of L'. Therefore, $L = L'$ again. (Exercise 5)

- Caution: One must not over-interpret the equality $Translation(L) = L$ (which is the same as $L = L'$).
- All the equality says is that the two lines L and L' coincide completely. It is easy (but wrong) to infer from the equality $Translation(L) = L$ that, for any point P on L, $Translation(P) = P$. Suppose the vector $\overrightarrow{AB}$ lying on L is not the zero vector (i.e., assume $A \neq B$). Trace the line L on a transparency to obtain a red line L, and now slide the transparency along $\overrightarrow{AB}$. Then the red line, as a line, coincides with the original L, but clearly every point on L has been moved by the slide (the translation). Indeed, as we saw in Example 1 of Lesson 2, $Translation(A) = B \neq A$. Therefore, the equality $L' = L$ only says that for any point C on L, $Translation(C)$ is also a point on L, but so long as $\overrightarrow{AB}$ is not a zero vector, $Translation(C) \neq C$.

MP.6

Note to Teacher: Strictly speaking, we have not completely proved $L = L'$ in either case. To explain this, let us *define* what it means for two geometric figures F and G to be equal, i.e., $F = G$: it means each point of F is also a point of G and, conversely, each point of G is also a point of F. In this light, all we have shown above is that every point C' of L' belongs to L, then Q is also a point of L' To show the latter, we have to show that this Q is equal to $T(P)$ for some P on L. This will then complete the reasoning.

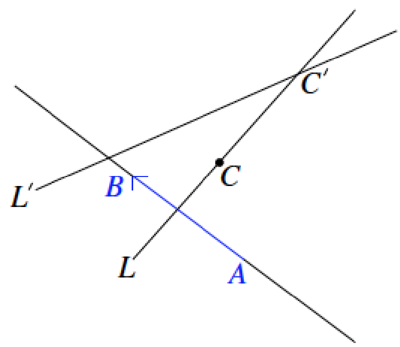
However, at this point of students' education in geometry, it may be prudent not to bring up such a sticky point because they already have their hands full with all the new ideas and new definitions. Simply allow the preceding reasoning to stand for now, and right the wrong later in the school year when students are more comfortable with the geometric environment.

- Next, if L is neither L_{AB} nor parallel to L_{AB}, then $L' \parallel L$

  If we use a transparency to see this translational image of L by the stated translation, then the pictorial evidence is clear: the line L moves in a parallel manner along $\overrightarrow{\mathrm{AB}}$, and a typical point C of L is translated to a point C' of L'. The fact that $L' \parallel L$ is unmistakable, as shown. In the classroom, students should be convinced by the pictorial evidence. If so, we will leave it at that. (Exercise 6)

MP.2 & MP.7

Note to Teacher: Here is a simple proof, but if you are going to present it in class, begin by asking students *how* they would prove that two lines are parallel. Make them see that *we have no tools in their possession to accomplish this goal*. It is only then that they see the need of invoking a proof by contradiction (see discussion in Lesson 3). *If there are no obvious ways to do something, then you just have to do the best you can by trying to see what happens if you assume the opposite is true*. Thus, if L' is not parallel to L, then they intersect at a point C'. Since C' lies on L', it follows from the definition of L' (as the image of L under the translation T) that there is a point C on L so that $T(C) = C'$.



It follows from B above that $L_{CC'} \parallel L_{AB}$. But both C and C' lie on L, so $L_{CC'} \parallel L$, and we get $L \parallel L_{AB}$. This contradicts the assumption that L is not parallel to L_{AB}, so that L could not possibly intersect L'. Therefore, $L' \parallel L$ after all.

- Note that a translation maps parallel lines to parallel lines. More precisely, consider a translation T along a vector $\overrightarrow{AB}$. Then:

  If L_1 and L_2 are parallel lines, so are $Translation(L_1)$ and $Translation(L_2)$.

  The reasoning is the same as before: copy L_1 and L_2 onto a transparency and then translate the transparency along $\overrightarrow{AB}$ If L_1 and L_2 do not intersect, then their red replicas on the transparency will not intersect either, no matter what $\overrightarrow{AB}$ is used. So $Translation(L_1)$ and $Translation(L_2)$ are parallel.

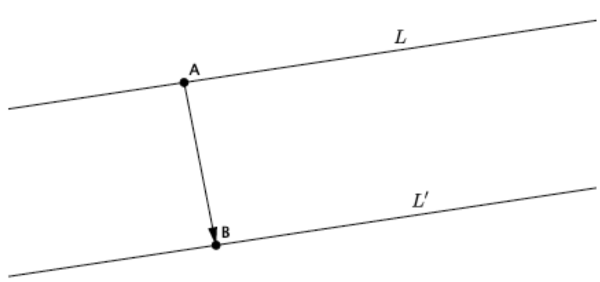
- We summarize these findings as follows:
  - Given a translation T along a vector $\overrightarrow{AB}$, let L be a line and let L' denote the image of L by T.
    - If $L \parallel L_{AB}$ or $L = L_{AB}$, then $L' \parallel L$
    - If L is neither parallel to L_{AB} nor equal to L_{AB} then $L' \parallel L$

MP.6

Exercises 5–6 (5 minutes)

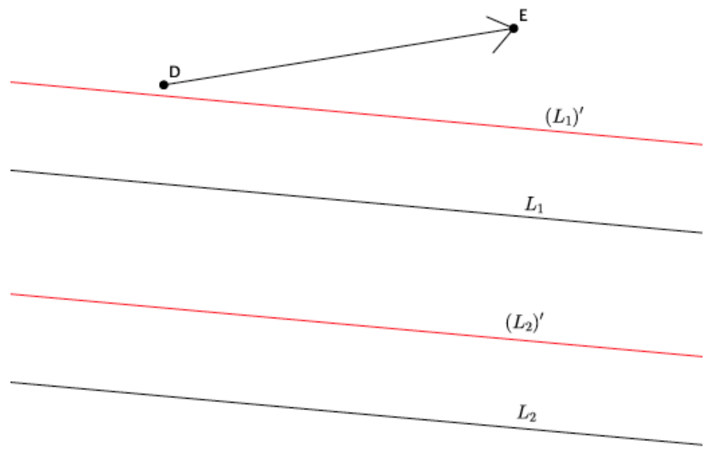
Students complete Exercises 5 and 6 in pairs or small groups.

5. Line L has been translated along vector $\overrightarrow{AB}$ resulting in L'. What do you know about lines L and L'?



$L \parallel T(L)$.

6. Translate L_1 and L_2 along vector $\overrightarrow{DE}$. Label the images of the lines. If lines L_1 and L_2 are parallel, what do you know about their translated images?



Since $L_1 \parallel L_2$ then $(L_1)' \parallel (L_2)'$

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- We know that there exists just one line, parallel to a given line and through a given point.
- We know that translations map parallel lines to parallel lines.
- Students know that when lines are translated they are either parallel to the given line, or the lines coincide.
- Students know something about the angles when lines are cut by a transversal (i.e., corresponding angles).

Lesson Summary

- **Two lines are said to be parallel if they do not intersect.**
- **Translations map parallel lines to parallel lines.**
- **Given a line L and a point P not lying on L, there is at most one line passing through P and parallel to L.**

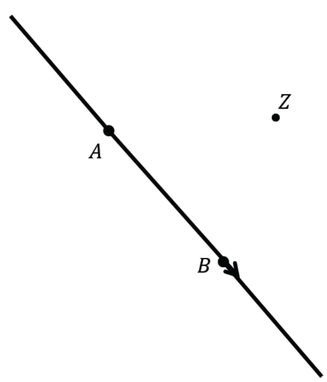
Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 3: Translating Lines

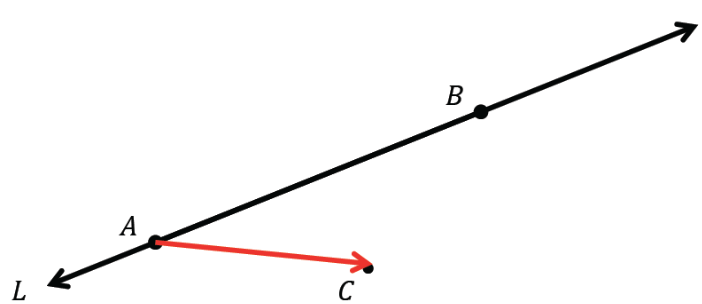
Exit Ticket

1. Translate point Z along vector $\overrightarrow{AB}$. What do you know about the line containing vector $\overrightarrow{AB}$ and the line formed when you connect Z to its image Z'?



2. Using the above diagram, what do you know about the lengths of segments ZZ' and AB?

3. Let points A and B be on line L, and the vector $\overrightarrow{AC}$ be given, as shown below. Translate line L along vector $\overrightarrow{AC}$. What do you know about line L and its image, L'? How many other lines can you draw through point C that have the same relationship as L and L'? How do you know?



Exit Ticket Sample Solutions

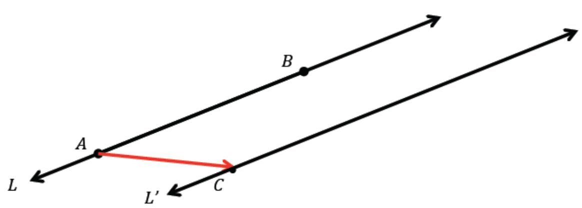
7. Translate point Z along vector $\overrightarrow{AB}$. What do you know about the line containing vector $\overrightarrow{AB}$ and the line formed when you connect Z to its image Z'?

   The line containing vector $\overrightarrow{AB}$ and ZZ' is parallel.

8. Using the above diagram, what do you know about the lengths of segment ZZ' and segment AB?

   The lengths are equal: $|ZZ'| = |AB|$.

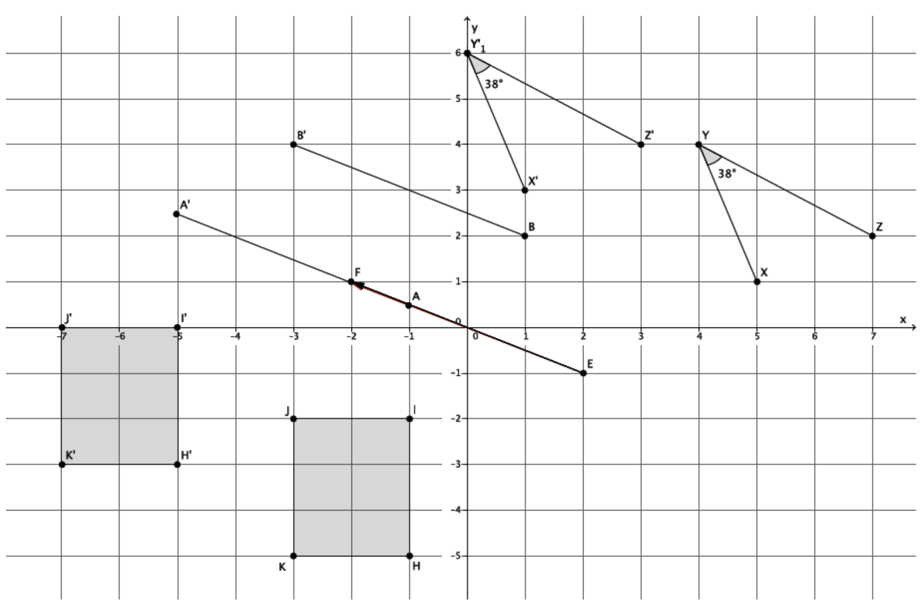
9. Let points A and B be on line L, and the vector $\overrightarrow{AC}$ be given, as shown below. Translate line L along vector $\overrightarrow{AC}$. What do you know about line L and its image, L'? How many other lines can you draw through point C that have the same relationship as L and L'? How do you know?



   L and L' are parallel. There is only one line parallel to line L that goes through point C. The fact that there is only one line through a point parallel to a given line guarantees it.

Problem Set Sample Solutions

1. Translate $\angle XYZ$, point A, point B, and rectangle $HIJK$ along vector $\overrightarrow{EF}$. Sketch the images and label all points using prime notation.



2. What is the measure of the translated image of $\angle XYZ$. How do you know?

   38°. Translations preserve angle measure.

3. Connect B to B'. What do you know about the line formed by BB' and the line containing the vector $\overrightarrow{EF}$?

   $BB' \parallel \overrightarrow{EF}$

4. Connect A to A'. What do you know about the line formed by AA' and the line containing the vector $\overrightarrow{EF}$?

   $AA' \parallel \overrightarrow{EF}$

5. Given that figure $HIJK$ is a rectangle, what do you know about lines HI and JK and their translated images? Explain.

   By definition of a rectangle, I know that $HI \parallel JK$. Since translations maps parallel lines to parallel lines, then $H'I' \parallel J'K'$.