Lesson 3: Numbers in Exponential Form Raised to a Power

Student Outcomes

- Students will know how to take powers of powers. Students will know that when a product is raised to a power, each factor of the product is raised to that power.
- Students will write simplified, equivalent numeric and symbolic expressions using this new knowledge of powers.

Classwork

Socratic Discussion (10 minutes)

Suppose we add 4 copies of 3, thereby getting $(3+3+3+3)$, and then add 5 copies of the sum. We get:

$$(3+3+3+3)+(3+3+3+3)+(3+3+3+3)+(3+3+3+3)+(3+3+3+3)$$

Now, by the definition of multiplication, adding 4 copies of 3 is denoted by (4×3), and adding 5 copies of this product is then denoted by $5\times(4\times 3)$. So,

$$5\times(4\times 3)=(3+3+3+3)+(3+3+3+3)+(3+3+3+3)+(3+3+3+3)+(3+3+3+3)$$

A closer examination of the right side of the above equation reveals that we are adding 3 to itself 20 times (i.e., adding 3 to itself (5×4) times). Therefore,

$$5\times(4\times 3)=(5\times 4)\times 3$$

MP.2 & MP.7

Now, replace repeated addition by repeated multiplication.

(For example, $(3\times 3\times 3\times 3)\times(3\times 3\times 3\times 3)\times\cdots\times(3\times 3\times 3\times 3)=3^4\times 3^4\times\cdots\times 3^4$.)

- What is multiplying 4 copies of 3, and then multiplying 5 copies of the product?
  - *Answer: Multiplying 4 copies of 3 is 3^4, and multiplying 5 copies of the product is $(3^4)^5$. We wish to say this is equal to 3^x for some positive integer x. By the analogy initiated in Lesson 1, the 5×4 in $(5\times 4)\times 3$ should correspond to the exponent x in 3^x, and therefore, the answer should be:*

$$(3^4)^5=3^{5\times 4}.$$

This is correct, because

$$\begin{aligned}(3^4)^5&=(3\times 3\times 3\times 3)^5\\&=\underbrace{(3\times 3\times 3\times 3)\times\cdots\times(3\times 3\times 3\times 3)}_{5\ times}\\&=\underbrace{3\times 3\times\cdots\times 3}_{5\times 4\ times}\\&=3^{5\times 4}\end{aligned}$$

Examples 1–2

Work through Examples 1 and 2 in the same manner as just shown (supplement with additional examples if needed).

Example 1

$$(7^2)^6 = (7 \times 7)^6$$
$$= \underbrace{(7 \times 7) \times \cdots \times (7 \times 7)}_{6\ times}$$
$$= \underbrace{7 \times \cdots \times 7}_{6 \times 2\ times}$$
$$= 7^{6 \times 2}$$

Example 2

$$(1.3^3)^{10} = (1.3 \times 1.3 \times 1.3)^{10}$$
$$= \underbrace{(1.3 \times 1.3 \times 1.3) \times \cdots \times (1.3 \times 1.3 \times 1.3)}_{10\ times}$$
$$= \underbrace{1.3 \times \cdots \times 1.3}_{10 \times 3\ times}$$
$$= 1.3^{10 \times 3}$$

In the same way, we have:

For any number x and any positive integers m and n,

$$(x^m)^n = x^{mn}$$

because

$$(x^m)^n = (\underbrace{x \cdot x \cdots x}_{m\ times})^n$$
$$= \underbrace{(x \cdot x \cdots x)}_{m\ times} \times \cdots \times \underbrace{(x \cdot x \cdots x)}_{m\ times} \quad (n\ times)$$
$$= x^{mn}$$

Exercises 1–6 (10 minutes)

Students complete Exercises 1–4 independently. Check answers, then, have students complete Exercises 5–6.

Exercise 1

$(15^3)^9 = 15^{9 \times 3}$

Exercise 2

$((-2)^5)^8 = (-2)^{8 \times 5}$

Exercise 3

$(3.4^{17})^4 = 3.4^{4 \times 17}$

Exercise 4

Let s be a number.

$(s^{17})^4 = s^{4 \times 17}$

Exercise 5

Sarah wrote that $(3^5)^7 = 3^{12}$. Correct her mistake. Write an exponential expression using a base of 3 and exponents of 5, 7, and 12 that would make her answer correct.

Correct way: $(3^5)^7 = 3^{35}$, ***Rewritten Problem:*** $3^5 \times 3^7 = 3^{5+7} = 3^{12}$

Exercise 6

A number y satisfies $y^{24} - 256 = 0$. What equation does the number $x = y^4$ satisfy?

Since $x = y^4$, then $(x)^6 = (y^4)^6$. Therefore, $x = y^4$ would satisfy the equation $x^6 - 256 = 0$.

Socratic Discussion (10 minutes)

MP.7

From the point of view of algebra and arithmetic, the most basic question about raising a number to a power has to be the following: How is this operation related to the four arithmetic operations? In other words, for two numbers x, y and a positive integer n,

1. How is $(xy)^n$ related to x^n and y^n?
2. How is $\left(\frac{x}{y}\right)^n$ related to x^n and y^n, $y \neq 0$?
3. How is $(x+y)^n$ related to x^n and y^n?
4. How is $(x-y)^n$ related to x^n and y^n?

The answers to the last two questions turn out to be complicated; students will learn about this in high school under the heading of the **binomial theorem**. However, they should at least be aware that, in general,

$(x+y)^n \neq x^n + y^n$, *unless* $n = 1$. *For example,* $(2+3)^2 \neq 2^2 + 3^2$.

Allow time for discussion of question 1. Students can begin by talking in partners or small groups, then share with the class.

Some students may want to simply multiply 5×8, but remind them to focus on the above stated goal which is to relate $(5 \times 8)^{17}$ to 5^{17} and 8^{17}. Therefore, we want to see 17 copies of 5 and 17 copies of 8 on the right side. Multiplying 5×8 would take us in a different direction.

> *Scaffolding:*
> - Provide a numeric example for students to work on: $(5 \times 8)^{17} = 5^{17} \times 8^{17}$

$$\begin{aligned}(5 \times 8)^{17} &= \underbrace{(5 \times 8) \times \cdots \times (5 \times 8)}_{17\ times} \\ &= \underbrace{(5 \times \cdots \times 5)}_{17\ times} \times \underbrace{(8 \times \cdots \times 8)}_{17\ times} \\ &= 5^{17} \times 8^{17}\end{aligned}$$

The following computation is a different way of proving the equality:

$$\begin{aligned}5^{17} \times 8^{17} &= \underbrace{(5 \times \cdots \times 5)}_{17\ times} \times \underbrace{(8 \times \cdots \times 8)}_{17\ times} \\ &= \underbrace{(5 \times 8) \times \cdots \times (5 \times 8)}_{17\ times} \\ &= (5 \times 8)^{17}\end{aligned}$$

Answer to Question 1:

Because in $(xy)^n$*, the factors* xy *will be repeatedly multiplied* n *times, resulting in factors of* x^n *and* y^n*:*

$$(xy)^n = x^n y^n$$

because

$(xy)^n = \underbrace{(xy)\cdots(xy)}_{n\ times}$ — By definition of raising a number to the n^{th} power

$= \underbrace{(x \cdot x \cdots x)}_{n\ times} \cdot \underbrace{(y \cdot y \cdots y)}_{n\ times}$ — By Commutative and Associative Properties

$= x^n y^n$ — By definition of x^n

For any numbers x and y, and positive integer n,

$$(xy)^n = x^n y^n$$

because

$$(xy)^n = \underbrace{(xy)\cdots(xy)}_{n\ times}$$
$$= \underbrace{(x \cdot x \cdots x)}_{n\ times} \cdot \underbrace{(y \cdot y \cdots y)}_{n\ times}$$
$$= x^n y^n$$

Exercises 7–13 (10 minutes)

Students complete Exercises 7–12 independently. Check answers.

Exercise 7

$(11 \times 4)^9 = 11^{1\times 9} \times 4^{1\times 9}$

Exercise 8

$(3^2 \times 7^4)^5 = 3^{5\times 2} \times 7^{5\times 4}$

Exercise 9

Let a, b, and c be numbers.

$(3^2 a^4)^5 = 3^{5\times 2} a^{5\times 4}$

Exercise 10

Let x be a number.

$(5x)^7 = 5^{1\times 7} \cdot x^{1\times 7}$

Exercise 11

Let x and y be numbers.

$(5xy^2)^7 = 5^{1\times 7} \cdot x^{1\times 7} \cdot y^{2\times 7}$

Exercise 12

Let a, b, and c be numbers.

$(a^2bc^3)^4 = a^{4\times 2} \cdot b^{4\times 1} \cdot c^{4\times 3}$

Next, have students work in pairs or small groups on Exercise 13 after you present the problem:

- Ask students to first explain why we must assume $y \neq 0$. Students should say that if the denominator were zero then the fraction would be undefined.
- *Answer: The answer to the fourth question is similar to the third: If x, y are any two numbers, such that $y \neq 0$ and n is a positive integer, then*

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

> *Scaffolding:*
> - Have students review problems just completed.
> - Remind students to begin with the definition of a number raised to a power.

Exercise 13

Let x and y be numbers, $y \neq 0$, and let n be a positive integer. How is $\left(\frac{x}{y}\right)^n$ related to x^n and y^n?

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

Because

$$\left(\frac{x}{y}\right)^n = \underbrace{\frac{x}{y} \times \cdots \times \frac{x}{y}}_{n \text{ times}} \qquad \textit{By definition}$$

$$= \frac{\overbrace{x \cdot x \cdots x}^{n \text{ times}}}{\underbrace{y \cdot y \cdots y}_{n \text{ times}}} \qquad \textit{By the Product Formula}$$

$$= \frac{x^n}{y^n} \qquad \textit{By definition}$$

Let the students know that this type of reasoning is required to prove facts in mathematics. They should always supply a reason for each step or at least know the reason the facts are connected. Further, it is important to keep in mind what we already know in order to figure out what we do not know. Students are required to write two proofs for homework that are extensions of the proofs they have done in class.

Closing (2 minutes)

- Summarize, or have students summarize the lesson.
- Students should state that they now know how to take powers of powers.

Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 3: Numbers in Exponential Form Raised to a Power

Exit Ticket

Write each answer as a simplified expression that is equivalent to the given one.

1. $(9^3)^6 =$

2. $(113^2 \times 37 \times 51^4)^3 =$

3. Let x, y, z be numbers. $(x^2yz^4)^3 =$

4. Let x, y, z be numbers and let m, n, p, q be positive integers. $(x^m y^n z^p)^q =$

5. $\frac{4^8}{5^8} =$

Exit Ticket Sample Solutions

Write each answer as a simplified expression that is equivalent to the given one.

1. $(9^3)^6 =$

   $(9^3)^6 = 9^{6\times 3} = 9^{18}$

2. $(113^2 \times 37 \times 51^4)^3 =$

   $(113^2 \times 37 \times 51^4)^3 = \left((113^2 \times 37) \times 51^4\right)^3$ *(associative law)*

   $= (113^2 \times 37)^3 \times (51^4)^3$ *(because $(xy)^n = x^n y^n$ for all numbers x, y)*

   $= (113^2)^3 \times 37^3 \times (51^4)^3$ *(because $(xy)^n = x^n y^n$ for all numbers x, y)*

   $= 113^6 \times 37^3 \times 51^{12}$ *(because $(x^m)^n = x^{mn}$ for all numbers x)*

3. Let x, y, z be numbers. $(x^2yz^4)^3 =$

   $(x^2yz^4)^3 = \left((x^2 \times y) \times z^4\right)^3$ *(associative law)*

   $= (x^2 \times y)^3 \times (z^4)^3$ *(because $(xy)^n = x^n y^n$ for all numbers x, y)*

   $= (x^2)^3 \times y^3 \times (z^4)^3$ *(because $(xy)^n = x^n y^n$ for all numbers x, y)*

   $= x^6 \times y^3 \times z^{12}$ *(because $(x^m)^n = x^{mn}$ for all numbers x)*

   $= x^6y^3z^{12}$

4. Let x, y, z be numbers and let m, n, p, q be positive integers. $(x^my^nz^p)^q =$

   $(x^my^nz^p)^q = \left((x^m \times y^n) \times z^p\right)^q$ *(associative law)*

   $= (x^m \times y^n)^q \times (z^p)^q$ *(because $(xy)^n = x^n y^n$ for all numbers x, y)*

   $= (x^m)^q \times (y^n)^q \times (z^p)^q$ *(because $(xy)^n = x^n y^n$ for all numbers x, y)*

   $= x^{mp} \times y^{nq} \times z^{pq}$ *(because $(x^m)^n = x^{mn}$ for all numbers x)*

   $= x^{mq}y^{nq}z^{pq}$

5. $\frac{4^8}{5^8} =$

   $\frac{4^8}{5^8} = \left(\frac{4}{5}\right)^8$

Problem Set Sample Solutions

1. Show (prove) in detail why $(2 \cdot 3 \cdot 4)^4 = 2^4 3^4 4^4$.

$(2 \cdot 3 \cdot 4)^4 = (2 \cdot 3 \cdot 4)(2 \cdot 3 \cdot 4)(2 \cdot 3 \cdot 4)(2 \cdot 3 \cdot 4)$	*By definition*
$= (2 \cdot 2 \cdot 2 \cdot 2)(3 \cdot 3 \cdot 3 \cdot 3)(4 \cdot 4 \cdot 4 \cdot 4)$	*By repeated use of the Commutative and Associative Properties*
$= 2^4 3^4 4^4$	*By definition*

2. Show (prove) in detail why $(xyz)^4 = x^4 y^4 z^4$ for any numbers x, y, z.

The left side of the equation, $(xyz)^4$, means $(xyz)\,(xyz)\,(xyz)\,(xyz)$. Using the Commutative and Associative Properties of Multiplication, we can write $(xyz)\,(xyz)\,(xyz)\,(xyz)$ as $(xxxx)(\,yyyy)\,(zzzz)$, which in turn can be written as $x^4 y^4 z^4$, which is what the right side of the equation states.

3. Show (prove) in detail why $(xyz)^n = x^n y^n z^n$ for any numbers x, y, z, and for any positive integer n.

The left side of the equation, $(xyz)^n$ means $\underbrace{(xyz) \cdot (xyz) \cdots (xyz)}_{n \text{ times}}$. Using the Commutative and Associative Properties of Multiplication, $\underbrace{(xyz) \cdot (xyz) \cdots (xyz)}_{n \text{ times}}$ can be rewritten as $\underbrace{(x \cdot x \cdots x)}_{n \text{ times}}\underbrace{(y \cdot y \cdots y)}_{n \text{ times}}\underbrace{(z \cdot z \cdots z)}_{n \text{ times}}$ and finally, $x^n y^n z^n$, which is what the right side of the equation states. We can also prove this equality by a different method, as follows. Beginning with the right side, $x^n y^n z^n$ means $\underbrace{(x \cdot x \cdots x)}_{n \text{ times}}\underbrace{(y \cdot y \cdots y)}_{n \text{ times}}\underbrace{(z \cdot z \cdots z)}_{n \text{ times}}$, which by the Commutative Property of Multiplication can be rewritten as $\underbrace{(xyz) \cdot (xyz) \cdots (xyz)}_{n \text{ times}}$. Using exponential notation, $\underbrace{(xyz) \cdot (xyz) \cdots (xyz)}_{n \text{ times}}$ can be rewritten as $(xyz)^n$, which is what the left side of the equation states.