



Lesson 20: Modeling Riverbeds with Polynomials

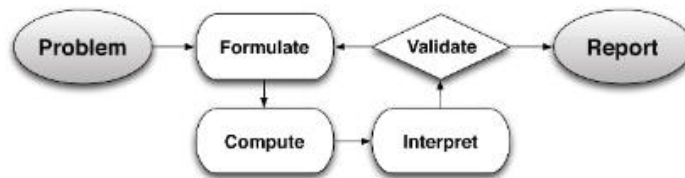
Student Outcomes

- Students learn to fit polynomial functions to data values.

Lesson Notes

In previous modeling lessons, students relied on the graphing calculator to find polynomial functions that fit a set of data. Lessons 20 and 21 comprise a two-lesson modeling exercise in which students model the shape of a riverbed in order to calculate flow rate of the river which can be used to determine if the river is vulnerable to flooding. In this first lesson, students use the Remainder Theorem as a way to fit a polynomial function to data points without relying on technology. In the second lesson, students will use technology, particularly Wolfram Alpha, to find the equation of the interpolating polynomial that best fits the data. The online calculations through Wolfram Alpha can be done as a demonstration for the class or individually if students have access to computers, or can be replaced by polynomial regression on a graphing calculator.

Students have some experience with modeling with polynomials from previous lessons in this module, but this is the first lesson sequence that takes them through the full modeling cycle as outlined in the modeling standard and as shown in the figure below. Students must first formulate a plan and develop a method for using the Remainder Theorem to fit a polynomial function to data. They must then use the model to compute the cross-sectional area of the riverbed and then volumetric flow of the water in the river. Students interpret their results and discuss how the results would be useful to the EPA.



Classwork

Mathematical Modeling Exercise (7 minutes): Discussion

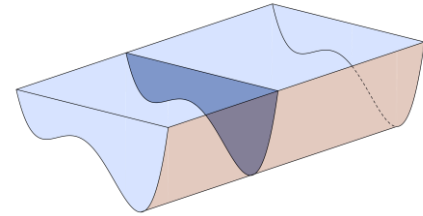
Have students read through the Mathematical Modeling Exercise. Have them discuss with a partner what questions need to be answered and brainstorm some ideas of how to answer them before having the discussion below. Tell the students that at the end of the next lesson, they will be expected to write a short report about the volumetric flow of the river. They will need to include their data and calculations in their report and should be preparing the report throughout the modeling exercise. Some pertinent vocabulary for this lesson includes the following:

Flow Rate (or volumetric flow rate): The volume of fluid that passes through a given surface per unit time.

Riverbed: The channel in which a river flows.

Cross-Section: A two-dimensional view of a slice through an object.

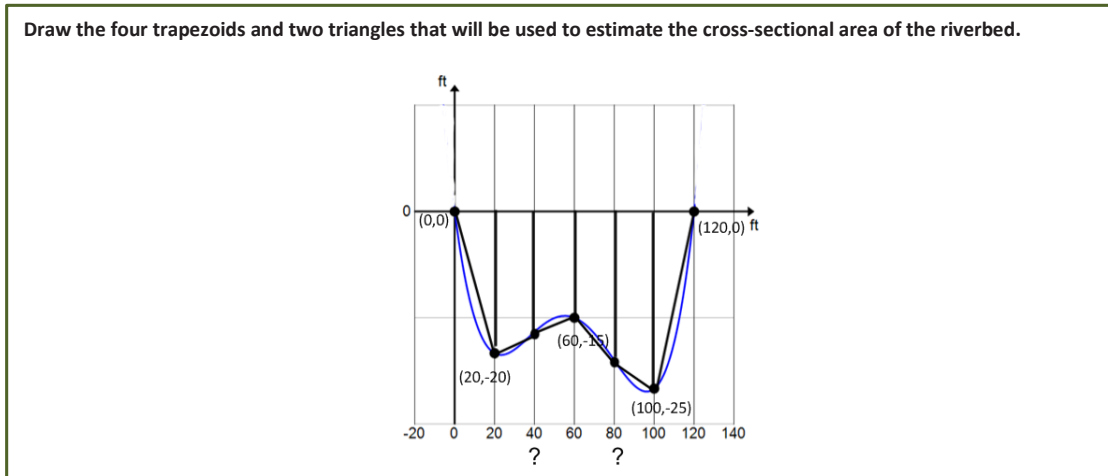
In order for students to formulate a plan for the modeling the shape of the riverbed, they must change their perspective from working with a three-dimensional idea of a river to the idea of working solely with a two-dimensional cross section. A figure such as the one to the right will help students see how the cross-sectional slice of the river relates to the river as a whole. The lower edge of the cross-section (shown in darker blue) is the curve that we need to find that goes through the given data points.



- Read through the statement of the Mathematical Modeling Exercise. What is our goal?
 - *We need to find the flow rate of water through the riverbed.*
- How can we estimate the flow rate of water?
 - *We need the area of the cross-section and the current speed of the water. Then the flow rate is the product of the cross-sectional area (in square feet) and the speed of the water (in feet per second).*
- To estimate the area of the cross-section, what information do we need?
 - *We need enough data points to make a good estimate.*
- To gather enough data points, we will need to make an assumption that the riverbed changes smoothly between the data points we have. With that assumption, we could estimate other data points by approximating the curve of the cross-section using a polynomial function.
- What is our first task?
 - *We need to find a polynomial function to fit the data.*
- Draw the graph of a polynomial function that passes through the 5 given data points.

Mathematical Modeling Exercise

The Environmental Protection Agency (EPA) is studying the flow of a river in order to establish flood zones. The EPA hired a surveying company to determine the flow rate of the river, measured as volume of water per minute. The firm set up a coordinate system and found the depths of the river at five locations as shown on the graph below. After studying the data, the firm decided to model the riverbed with a polynomial function and divide the area into six regions that are either trapezoidal or triangular so that the overall area can be easily estimated. The firm will need to approximate the depth of the river at two more data points in order to do this.



MP.1

- We have five data points. What information are we missing?
 - The depths at $x = 40$ and $x = 80$
- How can we find the missing values?
 - Find a polynomial function P that fits the given data and then use its equation to find $P(40)$ and $P(80)$, which should approximate the depths at $x = 40$ and $x = 80$.
- What is the lowest degree polynomial that could be used to model this data?
 - Degree four, because we have three relative maximum and minimum points.
- What constraints must our polynomial meet?
 - $P(0) = 0$, $P(20) = -20$, $P(60) = -15$, $P(100) = -25$, and $P(120) = 0$

Example 1 (12 minutes)

Work this example out two ways. First, let students find the equation by setting up a system of equations (as in Lesson 1). Then, demonstrate how the Remainder Theorem can be used to find the polynomial.

- Before trying to find a polynomial function whose graph goes through all five points, let's try a simpler problem and find a polynomial whose graph goes through just three of the points: $(0,28)$, $(2,0)$, and $(8,12)$.

Display these three points on a coordinate grid to use as reference through this discussion.

- Will the polynomial function that we find through these points be linear?
 - No. Looking at the graph we see that the points do not lie on a line.
- What do you think is the lowest degree polynomial we can find that will pass through the three points $(0,28)$, $(2,0)$, and $(8,12)$? (Remind students to look at the locations of these points in the plane as they decide on the lowest possible degree.)
 - It appears that we could fit a quadratic polynomial to these points; that is, they seem to lie on a parabola.
- Try to find a quadratic polynomial whose graph passes through these points. (Give students a hint to find a system of equations by plugging in x and $P(x)$ values into the equation $P(x) = ax^2 + bx + c$ as they did in Lesson 1.)

Allow students time to work and then discuss the process as a class.

Example 1

Find a polynomial P such that $P(0) = 28$, $P(2) = 0$, and $P(8) = 12$.

Since $P(0) = 28$, $c = 28 \rightarrow P(x) = ax^2 + bx + 28$

Since $P(2) = 0 \rightarrow 4a + 2b + 28 = 0$, so $2a + b = -14$ (1)

Since $P(8) = 12 \rightarrow 64a + 8b + 28 = 12$, so $8a + b = -2$ (2)

Subtract (1) from (2) $\rightarrow 6a = 12$, so $a = 2$

Substitute to find b . $\rightarrow 2(2) + b = -14$, so $b = -18$

$P(x) = 2x^2 - 18x + 28$

Have students check their equation by finding $P(0)$, $P(2)$, and $P(8)$ and showing that $P(0) = 28$, $P(2) = 0$, and $P(8) = 12$. Now work the problem again using the Remainder Theorem.

- We can also use the Remainder Theorem to find the quadratic polynomial P that satisfies $P(0) = 28$, $P(2) = 0$, and $P(8) = 12$. We have already seen a technique that will find this quadratic polynomial, but the method using the Remainder Theorem generalizes to higher degree polynomials.

- Use the first data point:

Since $P(0) = 28$, the Remainder Theorem tells us that for some polynomial a ,

$$P(x) = (x - 0)a(x) + 28$$

$$= x a(x) + 28.$$

- What do we know about the degree of the polynomial a ? How do we know?
 - Since P will be a quadratic polynomial, a must be a linear polynomial.

- Use the second data point:

Since $P(2) = 0$, we have

$$P(2) = 2 a(2) + 28 = 0$$

$$2 a(2) = -28$$

$$a(2) = -14.$$

Thus, the Remainder Theorem tells us that for some polynomial b ,

$$a(x) = (x - 2)b(x) - 14.$$

- What do we know about the degree of the polynomial b ? How do we know?
 - Since a is a linear polynomial, $b(x)$ must be a constant polynomial.
- Rewrite P in terms of b :

$$P(x) = x a(x) + 28$$

$$= x[(x - 2)b(x) - 14] + 28.$$

- Use the third data point:

Since $P(8) = 12$, we have

$$\begin{aligned} P(8) &= 8[(8 - 2)b(x) - 14] + 28 = 12 \\ 48b(x) - 104 + 28 &= 12 \\ 48b(x) &= 96 \\ b(x) &= 2. \end{aligned}$$

- Remember that we said the function $b(x)$ would be constant, and we found that $b(x) = 2$ for all values of x .
- Now we can rewrite the function P :

$$\begin{aligned} P(x) &= x[(x - 2)b(x) - 14] + 28 \\ &= x[(x - 2)2 - 14] + 28. \end{aligned}$$

- Do the three points satisfy this equation?
 - $P(0) = 0[(0 - 2) \cdot 2 - 14] + 28 = 28$ ✓
 - $P(2) = 2[(2 - 2) \cdot 2 - 14] + 28 = 0$ ✓
 - $P(8) = 8[(8 - 2) \cdot 2 - 14] + 28 = 12$ ✓
- For a final check, write the polynomial we just found in standard form.

$$\begin{aligned} P(x) &= x[(x - 2)2 - 14] + 28 \\ &= 2x^2 - 18x - 28. \end{aligned}$$
- What if we wanted to fit a polynomial function to two data points? What is the lowest degree of a polynomial that will fit two data points?
 - *We know from geometry that two points determine a unique line, so a polynomial of degree one will fit two data points.*
- In the next Example we will fit a polynomial to four data points. What is the lowest degree of a polynomial that will fit four data points?
 - *Since two data points can be fit by a degree 1 polynomial, and three data points can be fit by a degree 2 polynomial, it makes sense that four data points can be fit by a degree 3 polynomial.*

Example 2 (15 minutes)

The table below will help students organize their work as they proceed to find the cubic polynomial that fits four data points. Work through the first line of the table with the students, and then have the students work in groups to complete the table to determine the equation of the polynomial P . Point out to students that this would be a difficult task if we used systems of equations as we did first in Example 1. Have students present their work on the board.

MP.7

Example 2
Find a degree 3 polynomial P such that $P(-1) = -3$, $P(0) = -2$, $P(1) = -1$, and $P(2) = 6$.

Function Value	Substitute data point into the current form of the equation for P .	Apply the Remainder Theorem to a , b , or c	Rewrite the equation for P in terms of a , b or c .
$P(-1) = -3$			$P(x) = (x + 1)a(x) - 3$
$P(0) = -2$	$(0 + 1)a(0) - 3 = -2$	$a(x) = xb(x) + 1$	$P(x) = (x + 1)(xb(x) + 1) - 3$
$P(1) = -1$	$(1 + 1)(b(1) + 1) - 3 = -1$	$b(x) = (x - 1)c(x)$	$P(x) = (x + 1)(x(x - 1)c(x) + 1) - 3$
$P(2) = 6$	$(2 + 1)(2(2 - 1)c(2) + 1) - 3 = 6$	$c(x) = 1$	$P(x) = (x + 1)(x(x - 1) \cdot 1 + 1) - 3$

- Write the polynomial P in standard form.
 - $P(x) = (x + 1)(x^2 - x + 1) - 3$
 - $P(x) = x^3 - x^2 + x + x^2 - x + 1 - 3$
 - $P(x) = x^3 - 2$
- Verify that P satisfies the four given constraints.
 - $P(-1) = (-1)^3 - 2 = -3$
 - $P(0) = 0^3 - 2 = -2$
 - $P(1) = 1^3 - 2 = -1$
 - $P(2) = 2^3 - 2 = 6$

Closing (5 minutes)

- What methods have we used to find polynomial functions that fit given data?
 - *We plugged in data values to obtain a system of equations, and used the Remainder Theorem. In previous lessons, we used the graphing calculator.*
- If we want a polynomial to perfectly fit data points, how does the degree of the polynomial relate to the number of data points?
 - *The degree of the polynomial whose graph passes through a set of data points is one less than the number of data points.*

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

Lesson Summary

A linear polynomial is determined by 2 points on its graph.
 A degree 2 polynomial is determined by 3 points on its graph.
 A degree 3 polynomial is determined by 4 points on its graph.
 A degree 4 polynomial is determined by 5 points on its graph.
 The Remainder Theorem can be used to find a polynomial P whose graph will pass through a given set of points.

Exit Ticket (6 minutes)

Name _____

Date _____

Lesson 20: Modeling Riverbeds with Polynomials

Exit Ticket

Use the Remainder Theorem to find a quadratic polynomial P so that $P(1) = 5$, $P(2) = 12$, and $P(3) = 25$. Give your answer in standard form.

Exit Ticket Sample Solutions

Use the Remainder Theorem to find a quadratic polynomial P so that $P(1) = 5$, $P(2) = 12$, and $P(3) = 25$. Give your answer in standard form.

Since $P(1) = 5$, there is a linear polynomial a so that $P(x) = (x - 1)a(x) + 5$.

Since $P(2) = 12$, we have $P(2) = (2 - 1)a(2) + 5 = 12$, so $a(2) = 7$.

Because $a(2) = 7$, there exists a constant b so that $a(x) = (x - 2)b + 7$.

Substituting into $P(x)$ gives $P(x) = (x - 1)[(x - 2)b + 7]$.

Since $P(3) = 25$, we have $P(3) = (3 - 1)[(3 - 2)b + 7] = 25$. Solving for b gives $b = 3$.

It follows that $P(x) = (x - 1)[(x - 2)3 + 7]$, and in standard form we have $P(x) = 3x^2 - 2x + 4$.

Problem Set Sample Solutions

1. Suppose a polynomial P is such that $P(2) = 5$ and $P(3) = 12$.

a. What is the largest degree polynomial that can be uniquely determined given the information?

Degree one

b. Is this the only polynomial that satisfies $P(2) = 5$ and $P(3) = 12$?

No, there are an infinite number of polynomials that pass through those two points. However, two points will determine a unique equation for a linear function.

c. Use the Remainder Theorem to find the polynomial P of least degree that satisfies the two points given.

$$P(x) = 7x - 9$$

d. Verify that your equation is correct by demonstrating that it satisfies the given points.

$$P(2) = 7(2) - 9 = 5$$

$$P(3) = 7(3) - 9 = 12$$

2. Write a quadratic equation P such that $P(0) = -10$, $P(5) = 0$, and $P(7) = 18$ using the specified method.

a. Setting up a system of equations

$$P(x) = x^2 - 3x - 10$$

b. Using the Remainder Theorem

$$P(x) = x^2 - 3x - 10$$

3. Find a degree three polynomial function P such that $P(-1) = 0$, $P(0) = 2$, $P(2) = 12$, and $P(3) = 32$. Use the table below to organize your work. Write your answer in standard form, and verify by showing that each point satisfies the equation.

Function Value	Substitute data point into the current form of the equation for P .	Apply the Remainder Theorem to a , b , or c	Rewrite the equation for P in terms of a , b or c .
$P(-1) = 0$			$P(x) = (x + 1)a(x)$
$P(0) = 2$	$a(0) = 2$	$a(x) = xb(x) + 2$	$P(x) = (x + 1)(xb(x) + 2)$
$P(2) = 12$	$3(2b(2)) + 2 = 12$ $b(2) = 1$	$b(x) = (x - 2)c(x) + 1$	$P(x) = (x + 1)[x[(x - 2)c(x) + 1] + 2]$
$P(3) = 32$	$4[3(c(3) + 1)] + 2 = 32$ $c(3) = 1$	$c(3) = 1$	$P(x) = (x + 1)[x[(x - 2) + 1] + 2]$

$$P(x) = x^3 + x + 2$$

$$P(-1) = -1 - 1 + 2 = 0 \quad \checkmark$$

$$P(0) = 0 + 0 + 2 = 2 \quad \checkmark$$

$$P(2) = 8 + 2 + 2 = 12 \quad \checkmark$$

$$P(3) = 27 + 3 + 2 = 32 \quad \checkmark$$

4. The method used in Problem 3 is based on the Lagrange Interpolation method. Research Joseph-Louis Lagrange and write a paragraph about his mathematical work.

Answers will vary.