Lesson 16: Between-Figure and Within-Figure Ratios

Student Outcomes

- Students indirectly solve for measurements involving right triangles using scale factors, ratios between similar figures, and ratios within similar figures.
- Students use trigonometric ratios to solve applied problems.

Lesson Notes

At this point students are very familiar with how to use a scale factor with similar figures to determine unknown lengths of figures. The goal of this lesson is to show students that the values of the ratios of corresponding sides between figures can be rewritten, equivalently, as ratios of corresponding lengths within figures. The work foreshadows ratios related to trigonometry. Though sine, cosine, and tangent are not formally defined in this lesson, students are essentially using the ratios as a premise for formal treatment of trigonometric ratios in Topic E.

Classwork

Opening Exercise (2 minutes)

Opening Exercise

At a certain time of day, a 12 meter flagpole casts an 8 meter shadow. Write an equation that would allow you to find the height, h, of the tree that uses the length, s, of the tree's shadow.

$$\frac{12}{8} = \frac{h}{s}$$

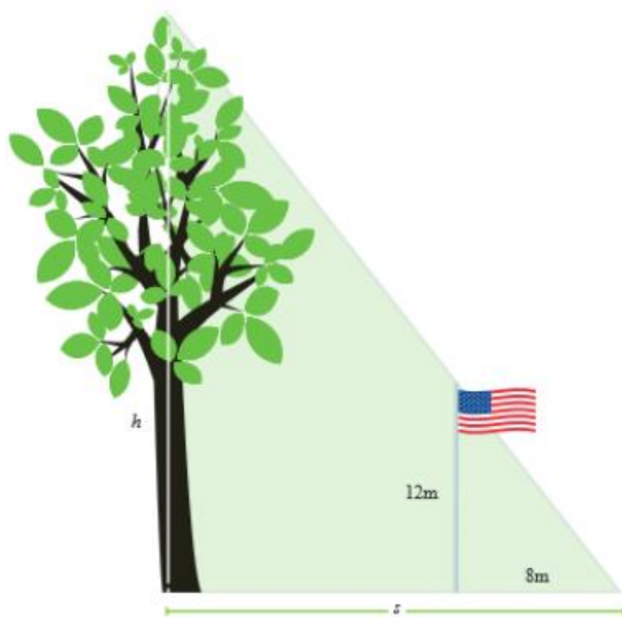
$$\frac{3}{2}s = h$$

OR

$$\frac{12}{h} = \frac{8}{s}$$

$$12s = 8h$$

$$\frac{3}{2}s = h$$

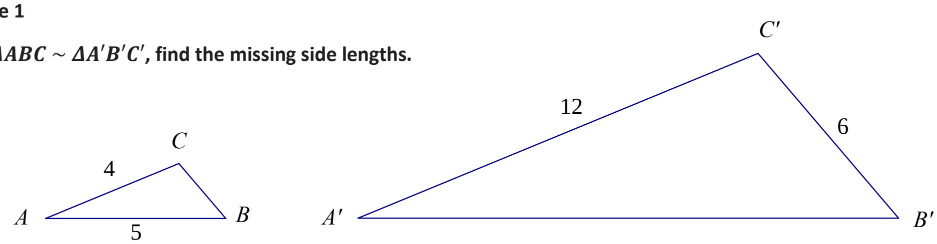


Example 1 (14 minutes)

The discussion following this exercise will highlight length relationships within figures. Many students will rely on ratios of lengths between figures to solve for unknown measurements of sides, rather than use ratios of lengths within a figure. The purpose of this example is to show students that comparing ratios between and within figures is mathematically equivalent.

Example 1

Given $\Delta ABC \sim \Delta A'B'C'$, find the missing side lengths.



- (Take a poll) How many people used the ratio of lengths $4:12$ or $12:4$ (the corresponding lengths of AC and $A'C'$) to determine the measurements of BC and $A'B'$?
- Are there any other length relationships we could use to set up an equation and solve for the missing side lengths?

MP.7

Provide students time to think about and try different equations that would lead to the same answer.

- *In addition to the ratio of corresponding lengths $AC:A'C'$, we can use a ratio of lengths within one of the triangles. For example, we could use the ratio of $AC:AB$ to find the length of $A'B'$. Equating the values of the ratios we get the following.*

$$\frac{4}{5} = \frac{12}{A'B'}$$
$$A'B' = 15$$

- *To find the length of BC, we can use the ratios $A'C':B'C'$ and $AC:BC$. Equating the values of the ratios we get the following.*

$$\frac{4}{BC} = \frac{12}{6}$$
$$BC = 2$$

- Why is it possible to use this length relationship, $AC:AB$, to solve for a missing side length?
  - *Take student responses and then offer the following explanation.*
- There are three methods that can be used to determine the missing side lengths.

  Method 1 (Scale Factor): Find the scale factor, and use it to compute for the desired side lengths.

  Since we know $AC = 4$ and the corresponding side $A'C' = 12$, the scale factor r satisfies $4r = 12$. So $r = 3$, $A'B' = rAB = 3 \cdot 5 = 15$, and $BC = \frac{B'C'}{r} = \frac{6}{3} = 2$.

  Method 2 (Ratios Between-Figures): Equate the values of the ratios of the corresponding sides.

  $\frac{A'B'}{AB} = \frac{A'C'}{AC}$ implies $\frac{A'B'}{5} = \frac{12}{4}$ and $A'B' = 15$. $\frac{BC}{B'C'} = \frac{AC}{A'C'}$ implies $\frac{BC}{6} = \frac{4}{12}$ and $BC = 2$.

Method 3 (Ratios Within-Figures): Equate the values of the ratios within each triangle.

$$\frac{A'B'}{A'C'} = \frac{AB}{AC} \text{ implies } \frac{A'B'}{12} = \frac{5}{4} \text{ and } A'B' = 15. \qquad \frac{BC}{AC} = \frac{B'C'}{A'C'} \text{ implies } \frac{BC}{4} = \frac{6}{12} \text{ and } BC = 2.$$

- Why does Method 3 work? That is, why can we use values of ratios *within* each triangle to find the missing side lengths?

Provide students time to explain why Method 3 works, and then offer the following explanation.

- If $\triangle ABC \sim \triangle A'B'C'$, then it is true that the ratio between any two side lengths of the first triangle is equal to the ratio between the corresponding side lengths of the second triangle; e.g., $AB: BC = A'B': B'C'$. This is because if r is the scale factor, then $AB: BC = rAB: rBC = A'B': B'C'$.

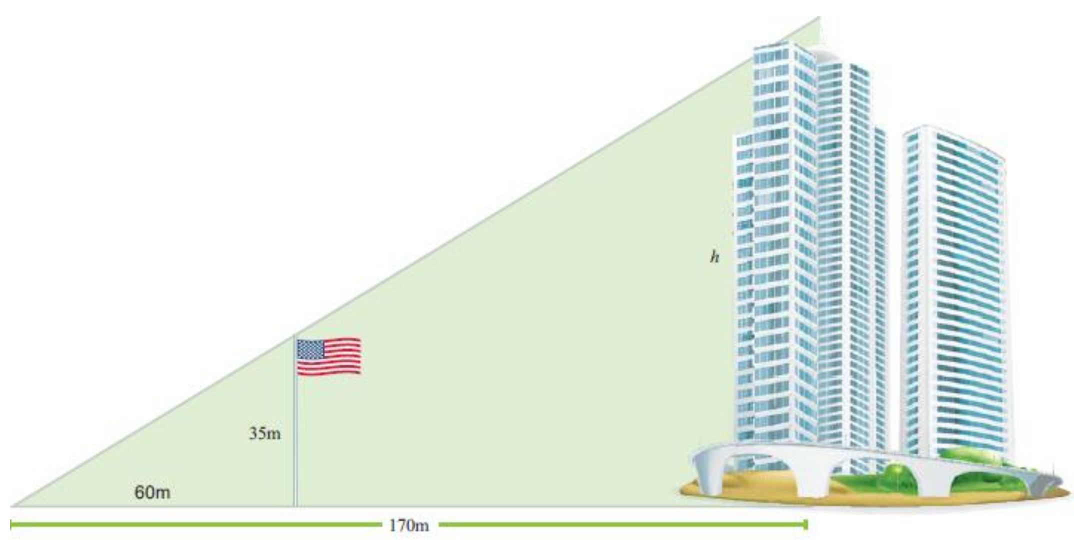
Revisit the Opening Exercise. Ask students to explain which method they used to write their equations for the height of the tree. Answers may vary, but students will likely have used Method 2 because that is what is most familiar to them.

Example 2 (7 minutes)

In this example, students use what they know about similar figures to indirectly measure the height of a building. If possible, use an image from your school or community to personalize the example.

- Suppose you want to know the height of a building. Would it make sense to climb to the roof and use a tape measure? Possibly. Suppose you want to know the distance between the Earth and the moon. Would a tape measure work in that situation? Not likely. For now, we will take indirect measurements of trees and buildings, but we will soon learn how the Greeks measured the distance from the Earth to the moon!

Example 2



In the diagram above a large flagpole stands outside of an office building. Marquis realizes that when he looks up from the ground, 60 m away from the flagpole, that the top of the flagpole and the top of the building line up. If the flagpole is 35 m tall, and Marquis is 170 m from the building, how tall is the building?

a. **Are the triangles in the diagram similar? Explain.**

Yes, the triangle formed by Marquis, the ground, and the flagpole is similar to the triangle formed by Marquis, the ground, and the building. They are similar by the AA criterion. Each of the triangles has a common angle with Marquis at the vertex, and each triangle has a right angle, where the flagpole and the building meet the ground.

b. **Determine the height of the building using what you know about scale factors.**

The scale factor is $\frac{170}{60} = \frac{17}{6} = 2.8\overline{3}$. *Then the height of the building is* $35(2.8\overline{3}) = 99.1\overline{6}$ *m.*

c. **Determine the height of the building using ratios *between* similar figures.**

$$\frac{35}{h} = \frac{60}{170}$$
$$5950 = 60h$$
$$h = 99.1\overline{6}$$

d. **Determine the height of the building using ratios *within* similar figures.**

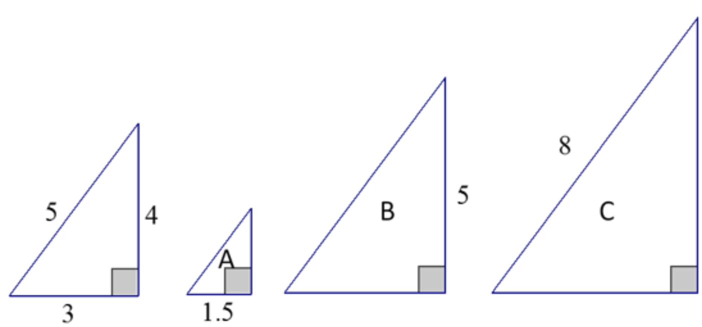
$$\frac{35}{60} = \frac{h}{170}$$
$$5950 = 60h$$
$$h = 99.1\overline{6}$$

Example 3 (12 minutes)

Work through the following example with the students. The goal is for them to recognize how the value of the ratio within a figure can be used to determine an unknown side length of a similar figure. Make clear to students that more than one ratio (from parts (a)–(c)) can be used to determine the requested unknown lengths in parts (d)–(f).

Example 3

The following right triangles are similar. We will determine the unknown side lengths by using ratios within the first triangle. For each of the triangles below, we define the base as the horizontal length of the triangle and the height as the vertical length.



Scaffolding:

- For some groups of students it may be beneficial to show only the 3-4-5 triangle as students complete parts (a)–(c) of the example.
- For advanced learners, have them consider why the triangle is called a 3-4-5 triangle and determine the significance of these numbers to any triangle that is similar to it.

a. Write and find the value of the ratio that compares the height to the hypotenuse of the leftmost triangle.

$$\frac{4}{5} = 0.8$$

b. Write and find the value of the ratio that compares the base to the hypotenuse of the leftmost triangle.

$$\frac{3}{5} = 0.6$$

c. Write and find the value of the ratio that compares the height to the base of the leftmost triangle.

$$\frac{4}{3} = 1.\overline{3}$$

d. Use the triangle with lengths $3–4–5$ and triangle A to answer the following questions.

i. Which ratio can be used to determine the height of triangle A?

The ratio that compares the height to the base can be used to determine the height of triangle A.

ii. Which ratio can be used to determine the hypotenuse of triangle A?

The ratio that compares the base to the hypotenuse can be used to determine the hypotenuse of triangle A.

iii. Find the unknown lengths of triangle A.

Let h represent the height of the triangle, then

$$\frac{h}{1.5} = 1.\overline{3}$$
$$h = 2.$$

Let c represent the length of the hypotenuse, then

$$\frac{1.5}{c} = 0.6$$
$$c = 2.5.$$

e. Use the triangle with lengths $3–4–5$ and triangle B to answer the following questions.

i. Which ratio can be used to determine the base of triangle B?

The ratio that compares the height to the base can be used to determine the base of triangle B.

ii. Which ratio can be used to determine the hypotenuse of triangle B?

The ratio that compares the height to the hypotenuse can be used to determine the hypotenuse of triangle B.

iii. **Find the unknown lengths of triangle B.**

Let b represent the length of the base, then

$$\frac{5}{b} = 1\frac{1}{3}$$
$$b = 3.75.$$

Let c represent the length of the hypotenuse, then

$$\frac{5}{c} = 0.8$$
$$c = 6.25.$$

f. **Use the triangle with lengths 3– 4– 5 and triangle C to answer the following questions.**

i. **Which ratio can be used to determine the height of triangle C?**

The ratio that compares the height to the hypotenuse can be used to determine the height of triangle C.

ii. **Which ratio can be used to determine the base of triangle C?**

The ratio that compares the base to the hypotenuse can be used to determine the base of triangle C.

iii. **Find the unknown lengths of triangle C.**

Let h represent the height of the triangle, then

$$\frac{h}{8} = 0.8$$
$$h = 6.4.$$

Let b represent the length of the base, then

$$\frac{b}{8} = 0.6$$
$$b = 4.8.$$

g. **Explain the relationship of the ratio of the corresponding sides within a figure to the ratio of corresponding sides within a similar figure.**

Corresponding lengths of similar figures have proportional lengths, but the ratio of two lengths within a figure is equal to the corresponding ratio of two lengths in a similar figure.

h. **How does the relationship you noted in part (g) allow you to determine the length of an unknown side of a triangle?**

This relationship allows us to find the length of an unknown side of a triangle. We can write the ratio of corresponding sides within the figure that contains the unknown side length. This ratio will be equal to the value of the ratio of corresponding sides where the lengths are known. Since the ratios are equal, the unknown length can be found by solving a simple equation.

Closing (5 minutes)

To close the lesson you may choose to ask the following three questions separately or have students complete a Quick Write for all three prompts and then discuss the three questions all at one time.

- What does it mean to use between-figure ratios of corresponding sides of similar triangles?
  - *Between-figure ratios are those where each ratio compares corresponding side lengths between two similar figures. That is, one number of the ratio comes from one triangle where the other number in the ratio comes from a different, similar triangle.*
- What does it mean to use within-figure ratios of corresponding sides of similar triangles?
  - *Within-figure ratios are those where each ratio is comprised of side lengths from one figure of the similar figures. That is, one ratio contains numbers that represent the side lengths from one triangle, and a second ratio contains numbers that represent the side lengths from a second, similar triangle.*
- How can within-figure ratios be used to find unknown side lengths of similar triangles?
  - *If a triangle is similar to another, then the within-figure ratios can be used to find the unknown lengths of the other triangle. For example, if the ratio that compares the height to the hypotenuse is known for one triangle and that ratio is equal to 0.6, then a ratio that compares the height to the hypotenuse of a similar triangle will also be equal to 0.6.*

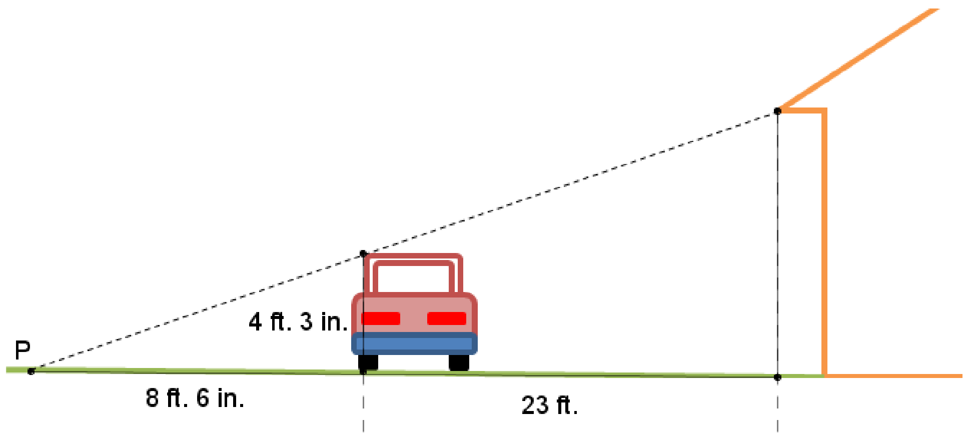
Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 16: Between-Figure and Within-Figure Ratios

Exit Ticket

Dennis needs to fix a leaky roof on his house but does not own a ladder. He thinks that a 25-foot ladder will be long enough to reach the roof, but he needs to be sure before he spends the money to buy one. He chooses a point P on the ground where he can visually align the roof of his car with the edge of the house roof. Help Dennis determine if a 25-foot ladder will be long enough for him to safely reach his roof.



Exit Ticket Sample Solutions

Dennis needs to fix a leaky roof on his house but does not own a ladder. He thinks that a 25-foot ladder will be long enough to reach the roof, but he needs to be sure before he spends the money to buy one. He chooses a point P on the ground where he can visually align the roof of his car with the edge of the house roof. Help Dennis determine if a 25-foot ladder will be long enough for him to safely reach his roof.

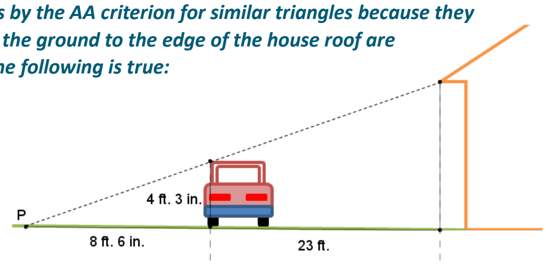
The height of the edge of the roof from the ground is unknown, so let x represent the distance from the ground to the edge of the roof. The nested triangles are similar triangles by the AA criterion for similar triangles because they share $\angle P$, and the height of the car and the distance from the ground to the edge of the house roof are both measured perpendicular to the ground. Therefore, the following is true:

$$\frac{8.5}{4.25} = \frac{(8.5 + 23)}{x}$$

$$8.5x = 133.85$$

$$x = 15.74705\ldots$$

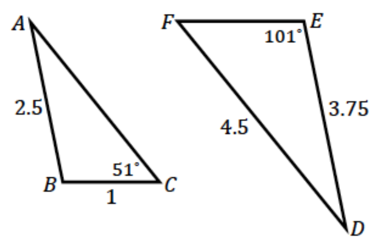
$$x \approx 15.75$$



The distance from the ground to the edge of the roof is 15.75 ft (or 15 ft. 9 in.). The 25-foot long ladder is clearly long enough for Dennis to safely reach his roof.

Problem Set Sample Solutions

1. **$\Delta DEF \sim \Delta ABC$. All side length measurements are in centimeters. Use between ratios and/or within ratios to determine the unknown side lengths.**



Using the given similarity statement, $\angle D$ corresponds with $\angle A$, and $\angle C$ corresponds with $\angle F$, so it follows that $\overline{AB}$ corresponds with $\overline{DE}$, $\overline{AC}$ with $\overline{DF}$, and $\overline{BC}$ with $\overline{EF}$.

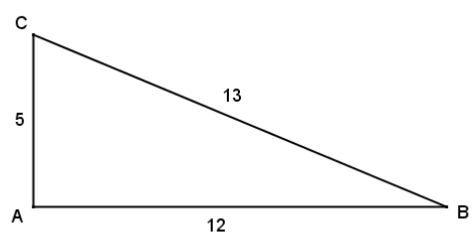
$$\frac{AB}{BC} = \frac{DE}{EF} \qquad \frac{AB}{DE} = \frac{AC}{DF}$$

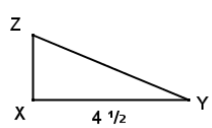
$$\frac{2.5}{1} = \frac{3.75}{EF} \qquad \frac{2.5}{3.75} = \frac{AC}{4.5}$$

$$2.5(EF) = 3.75 \qquad 3.75(AC) = 11.25$$

$$EF = 1.5 \qquad AC = 3$$

2. **Given $\Delta ABC \sim \Delta XYZ$, answer the following questions:**





   a. **Write and find the value of the ratio that compares the height $\overline{AC}$ to the hypotenuse of ΔABC.**

   $$\frac{5}{13}$$

   b. **Write and find the value of the ratio that compares the base $\overline{AB}$ to the hypotenuse of ΔABC.**

   $$\frac{12}{13}$$

c. Write and find the value of the ratio that compares the height $\overline{AC}$ to the base $\overline{AB}$ of ΔABC.

$$\frac{5}{12}$$

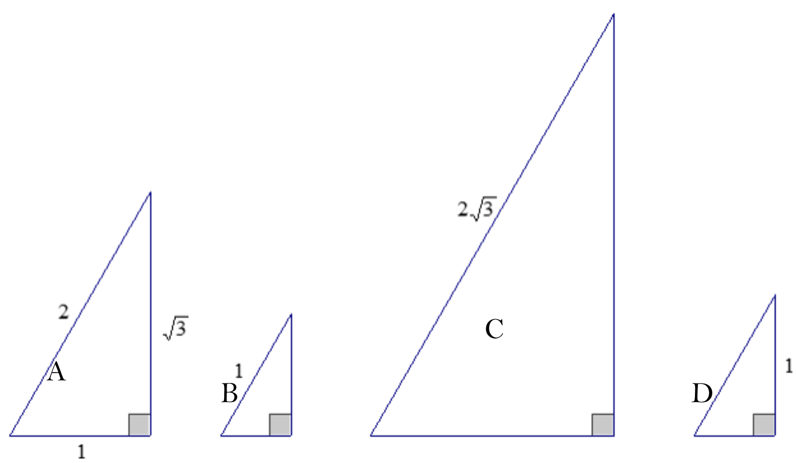
d. Use within-figure ratios to find the corresponding height of ΔXYZ.

$$\frac{5}{12} = \frac{XZ}{4.5}$$
$$XZ = 1\frac{7}{8}$$

e. Use within-figure ratios to find the hypotenuse of ΔXYZ.

$$\frac{12}{13} = \frac{4.5}{YZ}$$
$$YZ = 4\frac{7}{8}$$

3. Right triangles A, B, C, and D are similar. Determine the unknown side lengths of each triangle by using ratios of side lengths within triangle A.



a. Write and find the value of the ratio that compares the height to the hypotenuse of triangle A.

$$\frac{\sqrt{3}}{2} \approx 0.866$$

b. Write and find the value of the ratio that compares the base to the hypotenuse of triangle A.

$$\frac{1}{2} = 0.5$$

c. Write and find the value of the ratio that compares the height to the base of triangle A.

$$\frac{\sqrt{3}}{1} = \sqrt{3} \approx 1.73$$

d. Which ratio can be used to determine the height of triangle B? Find the height of triangle B.

The ratio that compares the height to the hypotenuse of triangle A can be used to determine the height of triangle B. Let the unknown height of triangle B be m.

$$\frac{\sqrt{3}}{2} = \frac{m}{1}$$

$$m = \frac{\sqrt{3}}{2}$$

e. Which ratio can be used to determine the base of triangle B? Find the base of triangle B.

The ratio that compares the base to the hypotenuse of triangle A can be used to determine the base of triangle B. Let the unknown base of triangle B be n.

$$\frac{1}{2} = \frac{n}{1}$$

$$n = \frac{1}{2}$$

f. Find the unknown lengths of triangle C.

The base of triangle C is $\sqrt{3}$.

The height of triangle C is 3.

g. Find the unknown lengths of triangle D.

The base of triangle D is $\frac{\sqrt{3}}{3}$.

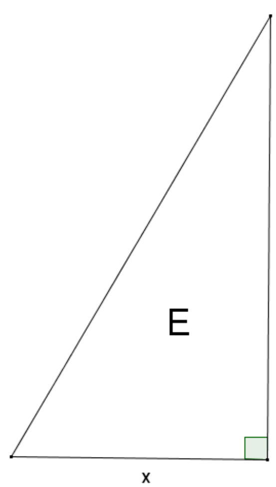
The hypotenuse of triangle D is $\frac{2\sqrt{3}}{3}$.

h. Triangle E is also similar to triangles A, B, C, and D. Find the lengths of the missing sides in terms of x.

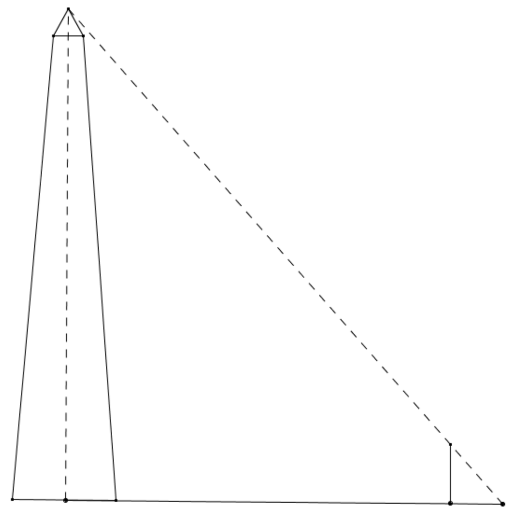
The base of triangle E is x.

The height of triangle E is $x\sqrt{3}$.

The hypotenuse of triangle E is $2x$.



4. Brian is photographing the Washington Monument and wonders how tall the monument is. Brian places his 5 ft. camera tripod approximately 100 yd. from the base of the monument. Lying on the ground, he visually aligns the top of his tripod with the top of the monument and marks his location on the ground approximately 2 ft. 9 in. from the center of his tripod. Use Brian's measurements to approximate the height of the Washington Monument.



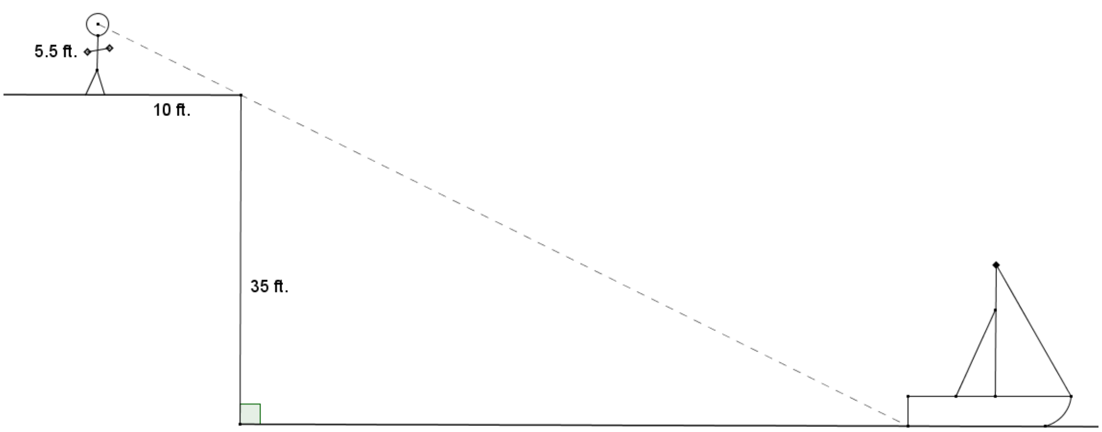
Brian's location on the ground is approximately 302.75 ft. from the base of the monument. His visual line forms two similar right triangles with the height of the monument and the height of his camera tripod.

$$\frac{5}{2.75} = \frac{h}{302.75}$$
$$1513.75 = 2.75h$$
$$550.5 \approx h$$

According to Brian's measurements, the height of the Washington Monument is approximately 550.5 ft.

Students may want to check the accuracy of this problem. The actual height of the Washington Monument is 555 ft.

5. Catarina's boat has come untied and floated away on the lake. She is standing atop a cliff that is 35 feet above the water in a lake. If she stands 10 feet from the edge of the cliff, she can visually align the top of the cliff with the water at the back of her boat. Her eye level is $5\frac{1}{2}$ feet above the ground. Approximately how far out from the cliff is Catarina's boat?

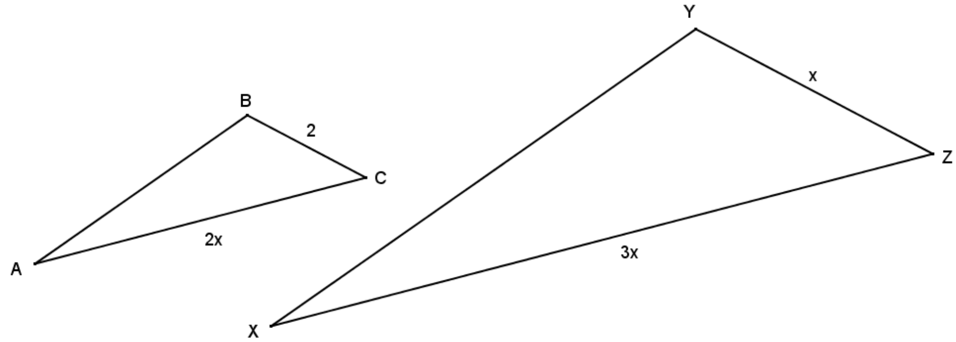


Catarina's height and the height of the cliff both are measured perpendicularly to the ground (and water), so both triangles formed by her visual path are right triangles. Assuming that the ground level on the cliff and the water are parallel, Catarina's visual path forms the same angle with the cliff as it does with the surface of the water (corr. ∠' s, parallel lines). So the right triangles are similar triangles by the AA criterion for showing triangle similarity, which means that the ratios within triangles will be equal. Let d represent the distance from the boat to the cliff:

$$\frac{5.5}{10} = \frac{35}{d}$$
$$5.5d = 350$$
$$d \approx 63.6$$

Catarina's boat is approximately 63.6 feet away from the cliff.

6. **Given the diagram below and $\triangle ABC \sim \triangle XYZ$, find the unknown lengths x, $2x$, and $3x$.**



The triangles are given as similar, so the values of the ratios of the sides within each triangle must be equal.

$$\frac{2}{2x} = \frac{x}{3x}$$

The sides of the larger triangle are unknown; however, the lengths include the same factor x, which is clearly non-zero, so the ratio of the sides must then be $\frac{1}{3}$. Similarly, the sides of the smaller triangle have the same factor 2, so the value of the ratio can be rewritten as $\frac{1}{x}$.

$$\frac{1}{x} = \frac{1}{3}$$

$$x = 3$$

Since the value of x is 3, it follows that $YZ = 3$, $AC = 6$, and $XZ = 9$.