



Lesson 6: Dilations as Transformations of the Plane

Student Outcomes

- Students review the properties of basic rigid motions.
- Students understand the properties of dilations and that a dilation is also an example of a transformation of the plane.

Lesson Notes

In Topic A, we plunged right into the *use* of dilations to create scale drawings and create arguments to prove the triangle side splitter theorem and dilation theorem. Topic B examines dilations in detail. In Grade 8 (Module 3), students discovered properties of dilations, such as that the dilation of a line maps onto another line or that the dilation of an angle maps onto another angle. We now examine how dilations differ from the other transformations and use reasoned arguments to confirm the properties of dilations that we observed in Grade 8.

We begin Topic B with a review of the rigid motions studied in Module 1 (Lessons 12–16).

Classwork

Discussion (7 minutes)

The goal of Lesson 6 is to study dilations as transformations of the plane. Begin with a review of what a transformation is and the category of transformations studied in Module 1. The following questions can be asked as part of a whole-group discussion, or, based on your judgment, you may want to ask for them to be written to let students express their thoughts on paper before discussing them aloud.

- Recall that our recent study of translations, reflections, and rotations was a study of transformations. With a partner, discuss what a *transformation of the plane* means.

Scaffolding:

Use Module 1, Lesson 17, Exercises 1–5 to provide students with good visuals and review the learned transformations and the conclusion that they are distance-preserving.

Allow students time to discuss before sharing out responses.

- *A transformation of the plane is a rule that assigns each point in the plane to a unique point. We use function notation to denote transformations, i.e., F denotes the transformation of a point, P , and is written as $F(P)$. Thus, a transformation moves point P to a unique point $F(P)$.*
- When we refer to the *image of P by F* , what does this refer to?
 - *The point $F(P)$ is called the image of P , or P' .*
 - *Recall that every point P' in the plane is the image of some point P , i.e., $F(P) = P'$.*
- In Module 1, we studied a set of transformations that we described as being “rigid”. What does the term rigid refer to?
 - *The transformations in Module 1—translations, reflections, and rotations—are all transformations of the plane that are rigid motions, or they are distance preserving. A transformation is distance-*

preserving if, given two points P and Q , the distance between these points is the same as the distance between the images of these points, that is the distance between $F(P)$ and $F(Q)$.

As we know, there are a few implied properties of any rigid transformation:

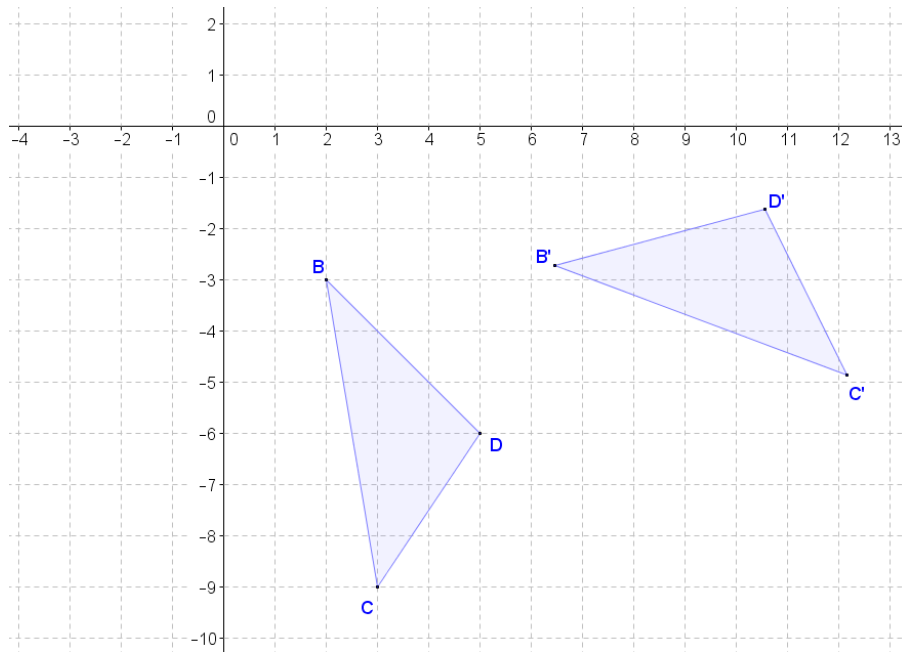
- a. Rigid motions map a line to a line, a ray to a ray, a segment to a segment, and an angle to an angle.
- b. Rigid motions preserve lengths of segments.
- c. Rigid motions preserve the measures of angles.

Exercises 1–6 (12 minutes)

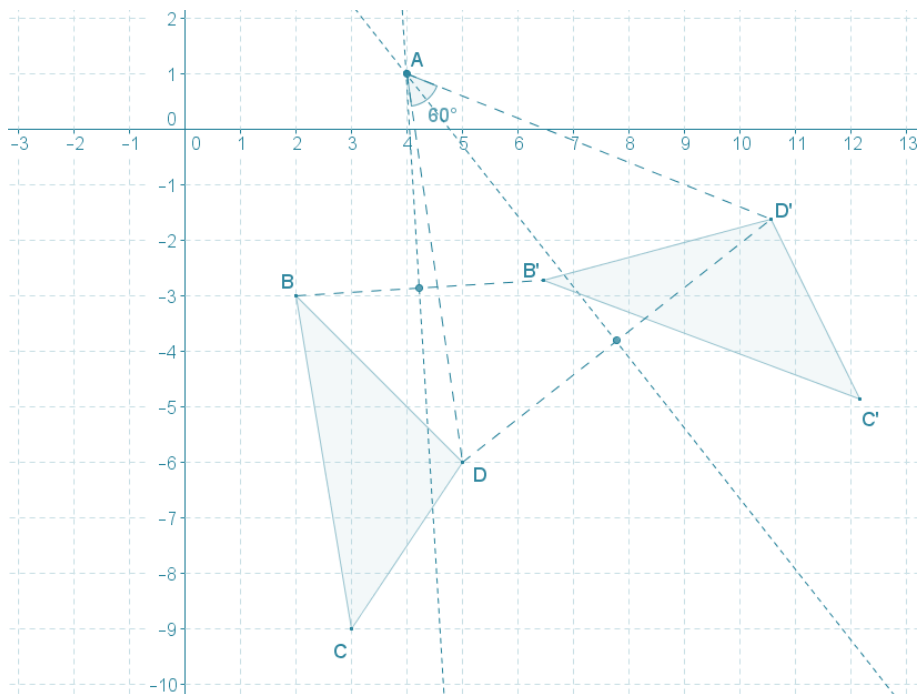
It is at the teacher’s discretion to assign only some or all of Exercises 1–6. Completion of all six exercises will likely require more than the allotted time.

<p>Exercises 1–6</p> <p>1. Find the center and the angle of the rotation that takes AB to $A'B'$. Find the image P' of point P under this rotation.</p>	
	<p><i>Step 1. Determine the location of center O as the intersection of the perpendicular bisectors of AA' and BB'</i></p>
<p><i>Step 2. Determine the angle of rotation by joining A to O and O to A'; the angle of rotation is 58° counterclockwise.</i></p>	<p><i>Step 3. Rotate P by 58°. P should remain a fixed distance away from O.</i></p>

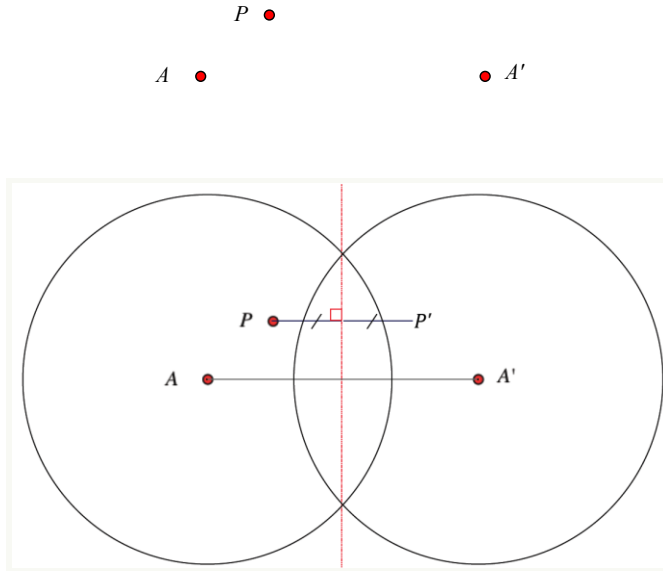
2. In the diagram below, $\triangle B'C'D'$ is the image of $\triangle BCD$ after a rotation about a point A . What are the coordinates of A , and what is the degree measure of the rotation?



By constructing the perpendicular bisector of each segment joining a point and its image, I found the center of dilation A to be $A(4, 1)$. Using a protractor, the angle of rotation from $\triangle BCD$ to $\triangle B'C'D'$ about point $A(4, 1)$ is 60° .

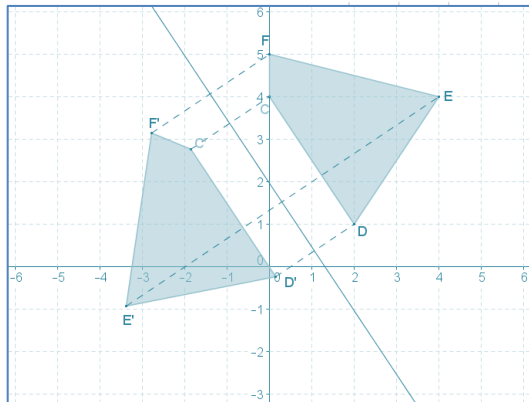
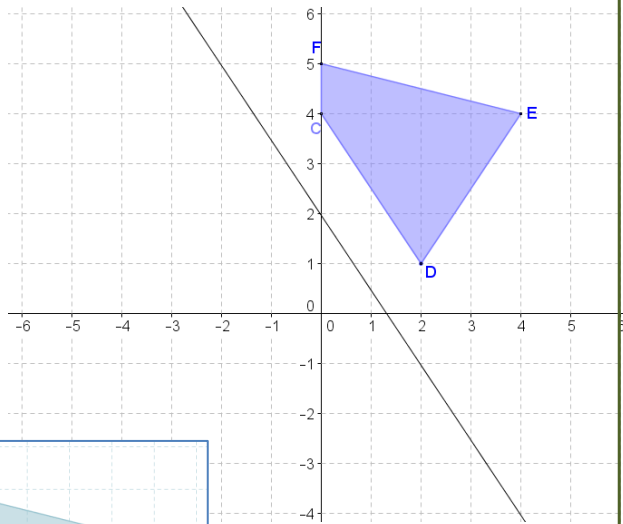


3. Find the line of reflection for the reflection that takes point A to point A' . Find the image P' under this reflection.

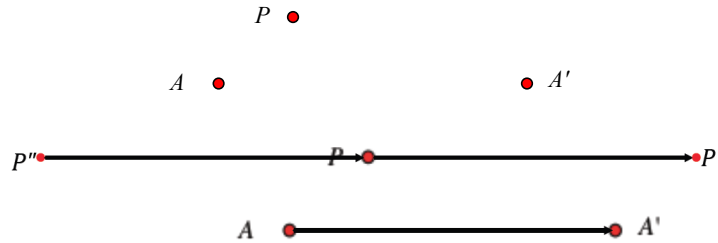


4. Quinn tells you that the vertices of the image of quadrilateral $CDEF$ reflected over the line representing the equation $y = -\frac{3}{2}x + 2$ are the following: $C'(-2, 3)$, $D'(0, 0)$, $E'(-3, -3)$, and $F'(-3, 3)$. Do you agree or disagree with Quinn? Explain.

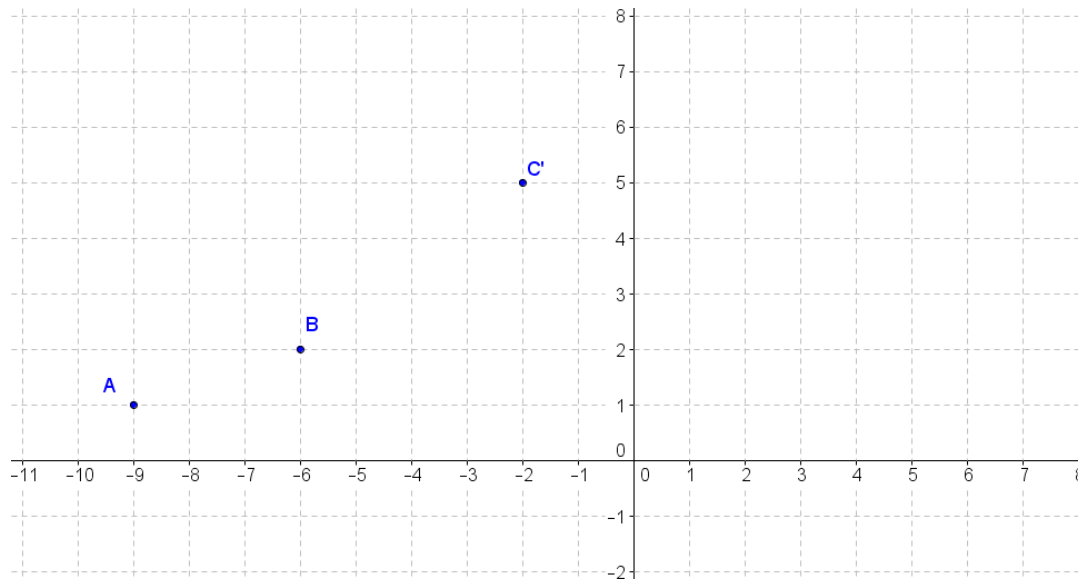
I disagree because under a reflection, an image point lies along a line through the pre-image point that is perpendicular to the line of reflection. The line representing the equation $y = \frac{2}{3}x + 4$ includes C and is perpendicular to the line of reflection, however, does not include the point $(-2, 3)$. Similar reasoning can be used to show that Quinn's coordinates for D' , E' , and F' are not the images of D , E , and F , respectively, under a reflection over $y = -\frac{3}{2}x + 2$.



5. A translation takes A to A' . Find the image P' and pre-image P'' of point P under this translation. Find a vector that describes the translation.



6. The point C' is the image of point C under a translation of the plane along a vector.



a. Find the coordinates of C if the vector used for the translation is \overrightarrow{BA} .

$C(1, 6)$

b. Find the coordinates of C if the vector used for the translation is \overrightarrow{AB} .

$C(-5, 4)$

Discussion (7 minutes)

Lead a discussion that highlights how dilations are like rigid motions and how they are different from them.

- In this module, we have used dilations to create scale drawings and to establish the triangle side splitter theorem and the dilation theorem. We pause here to inspect how dilations as a class of transformations are like rigid transformations and how they are different.

- What do dilations have in common with translations, reflections, and rotations?
 - All of these transformations meet the basic criteria of a transformation in the sense that each follows a rule assignment where any point P is assigned to a unique point $F(P)$ in the plane. Dilations and rotations both require a center in order to define the function rule.
- What distinguishes dilations from translations, reflections, and rotations?
 - Dilations are not a distance-preserving transformation like the rigid transformations. For every point P , other than the center, the point is assigned to $D_{O,r}(P)$, which is the point P on \overrightarrow{OP} so that the distance from $D_{O,r}(P)$ to O is $r \cdot OP$. The fact that distances are scaled means the transformation is not distance preserving.
- From our work in Grade 8, we have seen that dilations, just like the rigid motions, do in fact map segments to segments, lines to lines, and rays to rays, but we only confirmed this experimentally, and in the next several lessons, we will create formal arguments to prove why these properties hold true for dilations.
- One last feature that dilations share with the rigid motions is the existence of an inverse dilation, just as inverses exist for the rigid transformations. What this means is that composition of the dilation and its inverse takes every point in the plane to itself.
- Consider a 90° clockwise rotation about a center O : $R_{O,90}(P)$. The inverse rotation would be a 90° counter-clockwise rotation to bring the image of point P back to itself: $R_{O,-90}(R_{O,90}(P)) = R_{O,0}(P) = P$.
- What would an inverse dilation rely on to bring the image of a dilated point P' back to P ?
 - If we were dilating a point P by a factor of 2, the image would be written as $P' = D_{O,2}(P)$. In this case, P' is pushed away from the center O by a factor of two so that it is two times as far away from O . To bring it back to itself, we need to halve the distance or, in other words, scale by a factor of $\frac{1}{2}$, which is the reciprocal of the original scale factor: $D_{O,\frac{1}{2}}(D_{O,2}(P)) = D_{O,1}(P) = P$. Therefore, an inverse dilation will rely on the original center O but require a scale factor that is the reciprocal (or multiplicative inverse) of the original scale factor.

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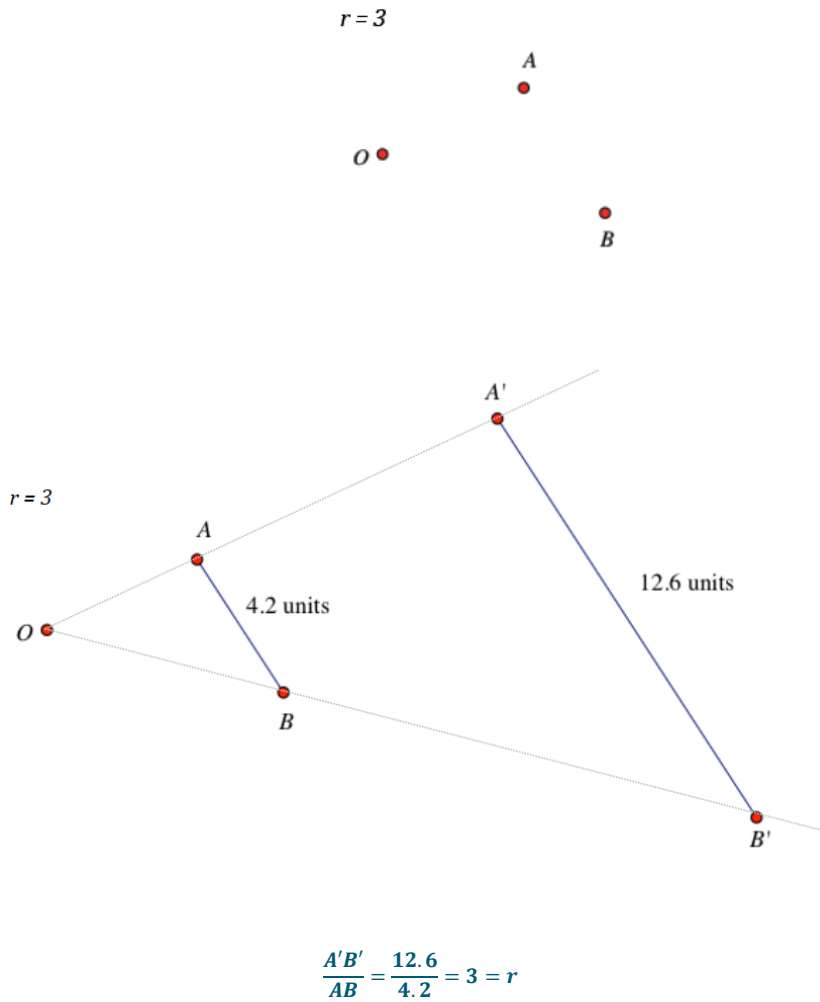
Exercises 7–9 (12 minutes)

Exercises 7–9

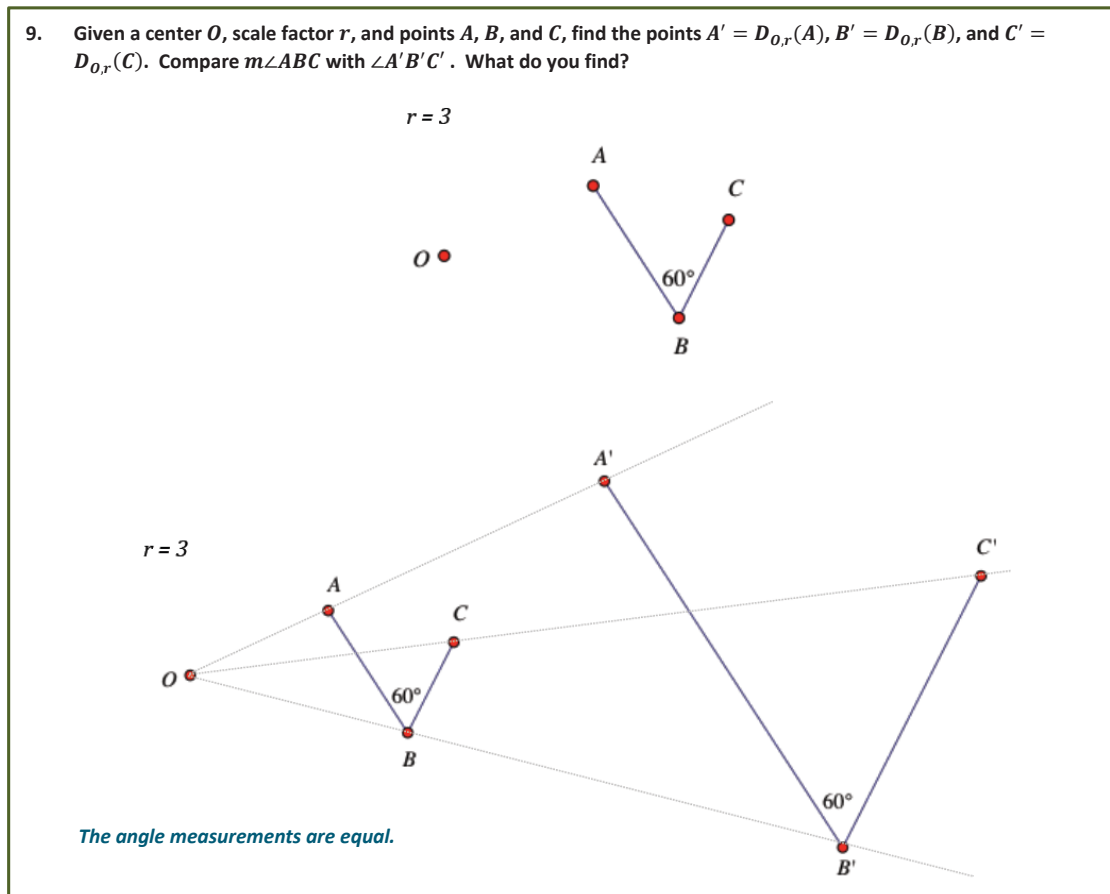
7. A dilation with center O and scale factor r takes A to A' and B to B' . Find the center O and estimate the scale factor r .

The estimated scale factor is $r \approx 2$.

8. Given a center O , scale factor r , and points A and B , find the points $A' = D_{O,r}(A)$ and $B' = D_{O,r}(B)$. Compare length AB with length $A'B'$ by division; in other words, find $\frac{A'B'}{AB}$. How does this number compare to r ?



9. Given a center O , scale factor r , and points A , B , and C , find the points $A' = D_{O,r}(A)$, $B' = D_{O,r}(B)$, and $C' = D_{O,r}(C)$. Compare $m\angle ABC$ with $\angle A'B'C'$. What do you find?



Closing (2 minutes)

- We have studied two major classes of transformations: those that are distance-preserving (translations, reflections, rotations) and those that are not (dilations).
- Like rigid motions, dilations involve a rule assignment for each point in the plane and also have inverse functions that return each dilated point back to itself.
- Though we have experimentally verified that dilations share properties similar to those of rigid motions, e.g., the property that lines map to lines, we have yet to establish these properties formally.

Lesson Summary

- There are two major classes of transformations; those that are distance-preserving (translations, reflections, rotations) and those that are not (dilations).
- Like rigid motions, dilations involve a rule assignment for each point in the plane and also have inverse functions that return each dilated point back to itself.

Exit Ticket (5 minutes)

Exit Ticket Sample Solutions

1. Which transformations of the plane are distance-preserving transformations? Provide an example of what this property means.

Rotations, translations, and reflections are distance-preserving transformations of the plane because for any two different points A and B in the plane, if F is a rotation, translation, or reflection that maps $A \rightarrow F(A)$ and $B \rightarrow F(B)$, $AB = F(A)F(B)$.

2. Which transformations of the plane preserve angle measure? Provide one example of what this property means.

Rotations, translations, reflections, and dilations all preserve angle measure. If lines \overline{AB} and \overline{BC} are coplanar and intersect at B to form $\angle ABC$ with measure n° , $\overline{F(A)F(B)}$ (or $\overline{A'B'}$) and $\overline{F(B)F(C)}$ (or $\overline{B'C'}$) intersect at $F(B)$ to form $\angle F(A)F(B)F(C)$ (or $\angle A'B'C'$) that also has measure n° .

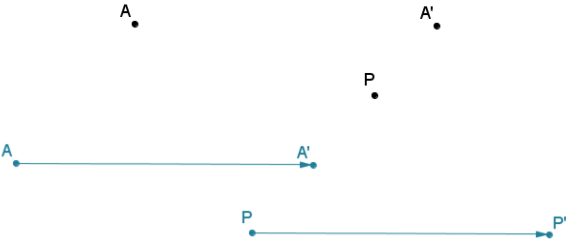
3. Which transformation is not considered a rigid motion and why?

A dilation is not considered a rigid motion because it does not preserve the distance between points. Under a dilation $D_{(0,r)}$ where $r \neq 1$, $D_{(0,r)}(A) = A'$ and $D_{(0,r)}(B) = B'$, $A'B' = r(AB)$, which means that $A'B'$ must have a length greater or less than AB .

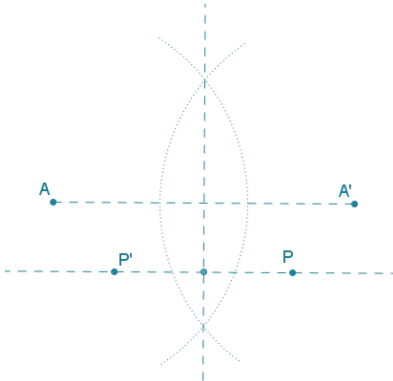
Problem Set Sample Solutions

1. In the diagram below, A' is the image of A under a single transformation of the plane. Use the given diagrams to show your solutions to parts (a)–(d).

a. Describe the translation that maps $A \rightarrow A'$, and then use the translation to locate P' , the image of P .

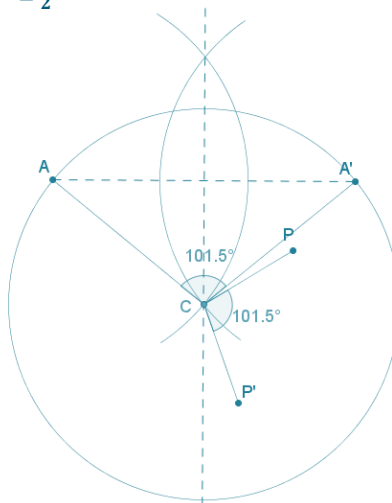


b. Describe the reflection that maps $A \rightarrow A'$, and then use the reflection to locate P' , the image of P .



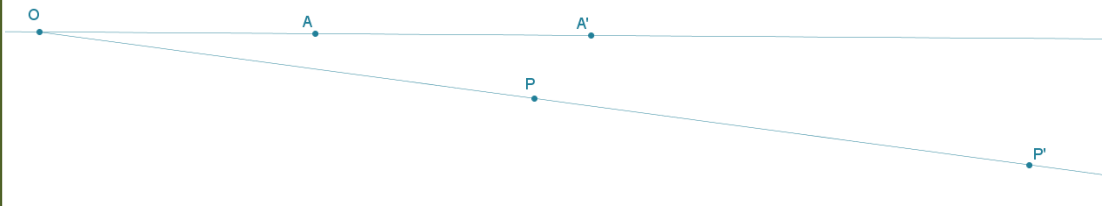
- c. Describe a rotation that maps $A \rightarrow A'$, and then use your rotation to locate P' , the image of P .

There are many possible correct answers to this part. The center of rotation C must be on the perpendicular bisector of $\overline{AA'}$ and the radius $CA \geq \frac{1}{2} AA'$.

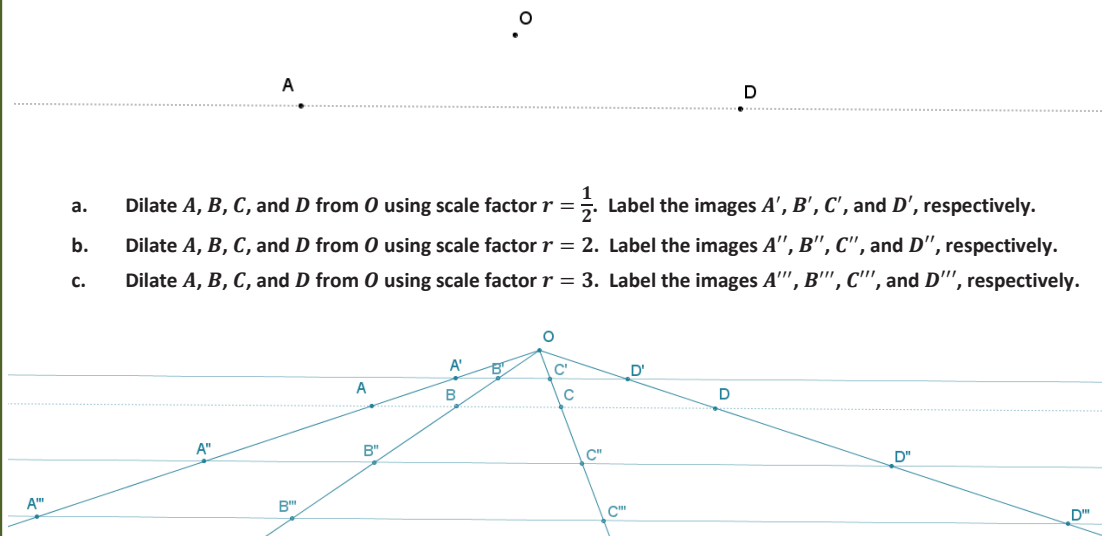


- d. Describe a dilation that maps $A \rightarrow A'$, and then use your dilation to locate P' , the image of P .

There are many possible correct answers to this part. The center of dilation must be on $\overline{AA'}$. If the scale factor chosen is $r > 1$, then A must be between O and A' . If the scale factor chosen is $r < 1$, then A' must be between A and O , and $OP' = r(OP)$. The sample shown below uses a scale factor $r = 2$.



2. On the diagram below, O is a center of dilation and \overline{AD} is a line not through O . Choose two points B and C on \overline{AD} between A and D .



- Dilate $A, B, C,$ and D from O using scale factor $r = \frac{1}{2}$. Label the images $A', B', C',$ and D' , respectively.
- Dilate $A, B, C,$ and D from O using scale factor $r = 2$. Label the images $A'', B'', C'',$ and D'' , respectively.
- Dilate $A, B, C,$ and D from O using scale factor $r = 3$. Label the images $A''', B''', C''',$ and D''' , respectively.

- Draw a conclusion about the effect of a dilation on a line segment based on the diagram that you drew. Explain.

Conclusion: Dilations map line segments to line segments.

3. Write the inverse transformation for each of the following so that the composition of the transformation with its inverse will map a point to itself on the plane.

a. $T_{\overline{AB}}$

The inverse of a translation along the vector \overline{AB} would be a translation along the vector \overline{BA} since this vector has the same magnitude but opposite direction. This translation will map any image point to its pre-image.

b. $r_{\overline{AB}}$

The inverse of a reflection over line AB is the same reflection. The points P and $r_{\overline{AB}}(P)$ are symmetric about \overline{AB} , so repeating the reflection takes a point back to itself.

c. $R_{C,45}$

The inverse of a 45° rotation about a point C would be a rotation about the same point C of -45° , the opposite rotational direction.

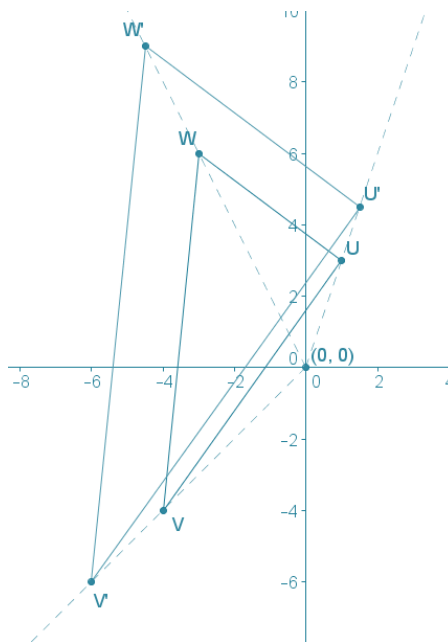
d. $D_{O,r}$

The inverse of a dilation with center O and scale factor r would be a dilation from center O with a scale factor of $\frac{1}{r}$. Point A in the plane is distance OA from the center of dilation O , and its image A' would, therefore, be at a distance $r(OA)$ from O . A dilation of A' with scale factor $\frac{1}{r}$ would map the A' to a point that is a distance $\frac{1}{r}(OA') = \frac{1}{r}(r(OA)) = 1(OA) = OA$. By the definition of a dilation, points and their images lie on the same ray that originates at the center of dilation. There is only one point on that ray at a distance OA from O , which is A .

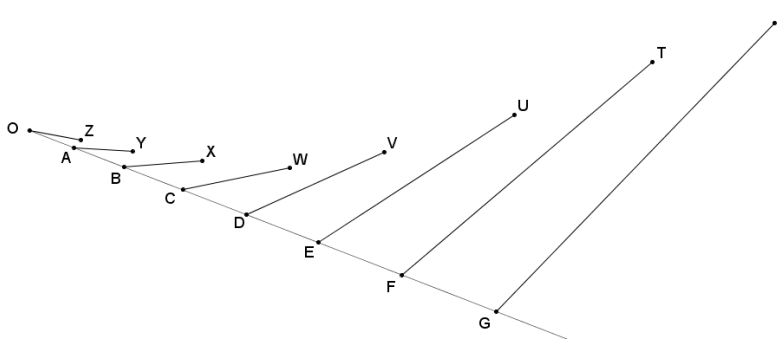
To the teacher: Problem 4 reviews the application of dilation on the coordinate plane that was studied in depth in Grade 8.

4. Given $U(1, 3)$, $V(-4, -4)$, and $W(-3, 6)$ on the coordinate plane, perform a dilation of $\triangle UVW$ from center $O(0, 0)$ with a scale factor of $\frac{3}{2}$. Determine the coordinates of images of points U , V , and W , and describe how the coordinates of the image points are related to the coordinates of the pre-image points.

Under the given dilation, U , V , and W map to U' , V' , and W' respectively. $U' = (\frac{3}{2}, \frac{9}{2})$, $V' = (-6, -6)$, and $W' = (-\frac{9}{2}, 9)$. For each point (X, Y) on the coordinate plane, its image point is $(\frac{3}{2}X, \frac{3}{2}Y)$ under the dilation from the origin with scale factor $\frac{3}{2}$.

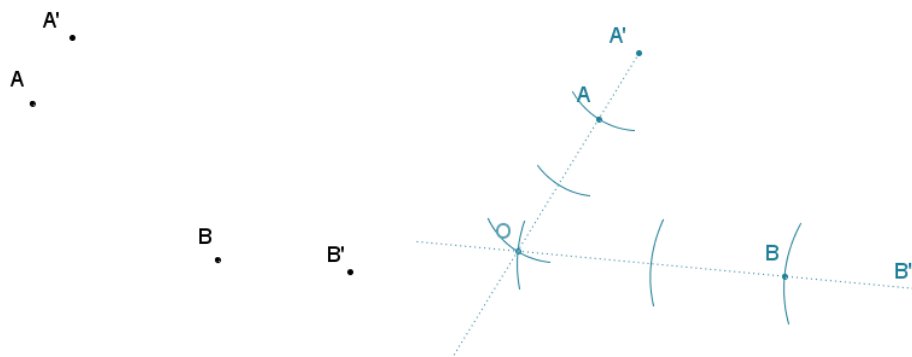


5. Points B, C, D, E, F , and G are dilated images of A from center O with scale factors 2, 3, 4, 5, 6, and 7, respectively. Are points Y, X, W, V, U, T , and S all dilated images of Z under the same respective scale factors? Explain why or why not.



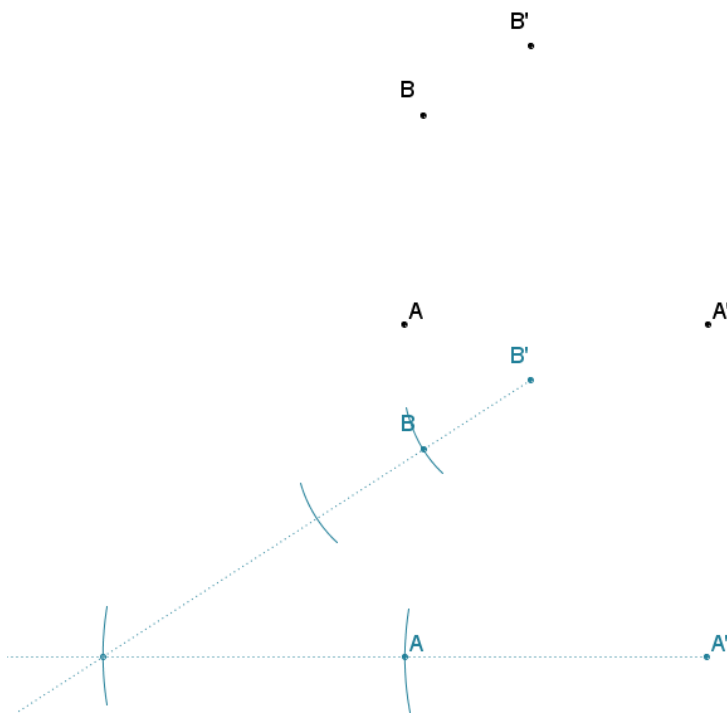
If points Y, X, W, V, U, T , and S were dilated images of Z , the images would all be collinear with O and Z ; however, the points are not all on a line, so they cannot all be images of point Z from center O . We also know that dilations preserve angle measures, and it is clear that each segment meets \overline{OA} at a different angle.

6. Find the center and scale factor that takes A to A' and B to B' , if a dilation exists.



The center of dilation is O , and the scale factor is $\frac{3}{2}$.

7. Find the center and scale factor that takes A to A' and B to B' , if a dilation exists.



After drawing $\overline{B'B}$ and $\overline{A'A}$, the rays converge at a supposed center; however, the corresponding distances are not proportional since $\frac{OA'}{OA} = 2$ and $\frac{OB'}{OB} \neq 2$. Therefore, a dilation does not exist that maps $A \rightarrow A'$ and $B \rightarrow B'$.

It also could be shown that \overline{AB} and $\overline{A'B'}$ are not parallel; therefore, the lengths are not proportional by the triangle side splitter theorem, and there is no dilation.